Controllability, Stabilizability, and Continuous-Time Markovian Jump Linear Quadratic Control

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Abstract—**This paper is concerned with the control of continuous-time linear systems that possess randomly jumping parameters which can be described by finite-state Markov processes. The relationship between appropriately defined controllability, stabilizability properties and the solution of the infinite time jump linear quadratic (JLQ) optimal control problem is also examined. Although the solution of the continuous-time Markov JLQ problem with finite or infinite time horizons is known, only sufficient conditions for the existence of finite cost, constant, stabilizing controls for the infinite time problem appear in the literature. In this paper, *necessary and sufficient* conditions are established. These conditions are based upon new definitions of controllability, observability, stabilizability, and detectability that are appropriate for continuous-time Markovian jump linear systems. These definitions play the same role for the JLQ problem as the deterministic properties do for the linear quadratic regulator (LQR) problem.**

I. Introduction and Problem Formulation

Fault-prone dynamic systems may experience abrupt changes in their structure and parameters, caused by phenomena such as component failures or repairs, changing subsystem interconnections, and abrupt environmental disturbances. Such systems can be modeled as operating in different "forms" [8], where each form corresponds to some combination of these events. We consider continuous-time linear systems with Markovian jumps, modeled by

$$\dot{x}(t) = A(r(t))x(t) + B(r(t))u(t) \tag{1}$$

where $t \in [t_0, T]$, T may be finite or infinite, $x(t) \in \mathbb{R}^n$ is the *x-process* state, $u(t) \in \mathbb{R}^m$ is the *x-process input*, and $A(t, r(t))$ and $B(t, r(t))$ are appropriately dimensioned real valued matrices. These matrices are functions of random process $\{r(t)\}$. This *form process* $\{r(t)\}$ is a continuous-time discrete-state Markov process taking values in a finite set $\mathbb{S} = \{1, 2, \cdots, s\}$ with transition probability matrix $P \triangleq \{p_{ij}\}$ given by

$$p_{ij} = \Pr\,(r(t+\Delta) = j \mid r(t) = i) = \begin{cases} \lambda_{ij}\Delta + 0(\Delta) & \text{if } i = j \\ 1 + \lambda_{ii}\Delta + 0(\Delta) & \text{if } i \neq j \end{cases} \tag{2}$$

where $\Delta > 0$. Here $\lambda_{ij} \geq 0$ is the form transition rate from i to $j (i \neq j)$, and

$$\lambda_i \triangleq -\lambda_{ii} = \sum_{j=1, j\neq i}^{s} \lambda_{ij} \tag{3}$$

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as in [29]. Note that $|\lambda_{ij}| < \infty$ because $r(t)$ is a finite state Markov process [10, pp. 150–151]. The system (1)–(3) is linear in x and u, for any given form process sample path. Note that, in general, A, B and the λ_{ij}'s could be explicit functions of time, as long as certain smoothness conditions are met. Let the initial values x_0 and r_0 be independent random variables; x_0 is also independent of the σ-algebra generated by $\{r(t), t \in (t_0, T]\}$. By [29], since almost all sample paths of $r(t)$ are constant except for a finite number of simple jumps in any finite subinterval of $[t_0, T\}$, we can define the paths of $x(t)$ by joining solution arcs of (1) at jump points of r. The $x(t)$ sample functions so determined are then continuous with probability one.

Subject to (1)–(3), we consider the minimization of

$$J(t_0, x(t_0), r(t_0), T, u) = E\left\{\int_{t_0}^{T} \left[x'(t)Q(r(t))x(t) + u'(t)R(r(t))u(t)\right] dt \,|\, x(t_0), r(t_0)\right\} \tag{4}$$

over form-dependent control laws $\psi \in \Psi$

$$u(t) = \psi(t, x(t), r(t)) \qquad \psi: [t_0, T] \times \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^m$$

where, for some constant k (depending on ψ),

$$\|\psi(t, x, r) - \psi(t, \tilde{x}, r)\| < k\|x - \tilde{x}\|, \qquad \psi(t, x, r) < k(1 + \|x\|)$$

for all t, x, $\tilde{x}$, r. Note that this growth condition rules out the use of impulse controls. Matrices R and Q are real valued and symmetric with $R(r(t))$ positive definite and $Q(r(t))$ positive semidefinite. Here $\|x\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$ is the Euclidean norm of vector x, and $\|A\|$ (the largest eigenvalue of $(A'A)^{1/2}$) is the corresponding operator norm of matrix A. We adopt $\psi \in \Psi$, with Ψ being the class of admissible controls as in [29]. Then the joint process $(x(t), r(t): t_0 \leq t \leq T)$ is a Markov process. We are assuming here that the values of $r(t)$ and $x(t)$ are available at time t, without error. The expectation in (4) is over the joint process $\{x, r\}$. The assumption that the form process value $r(t)$ is available at time t is not realistic, although in some applications measured values of $r(t)$ may be available with a small delay. In this paper, this perfect form observation assumption is crucial, since it allows us to avoid the "dual control" problem. The existence and applicability of a "separation" theorem for unobserved $r(t)$ are open questions.

In general, systems described by (1)–(3) belong to the category of "hybrid systems," since they combine a part of the state that takes values continuously ($x \in \mathbb{R}^n$) and another part of the state that takes discrete values ($r \in \mathbb{S}$). Such hybrid systems have been considered for the modeling of electric power systems by Willsky and Levy [27], for the control of a solar thermal central receiver by Sworder and Rogers [26]. Athans [2] suggested that this model setting also has the potential to become a basic framework in posing and solving control-related issues in Battle Management Command, Control, and Communications (BM/C[3]) systems.

The study of this continuous-time Markovian JLQ problem can be traced back (at least) to the work of Krasovskii and Lidskii

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[16]. Later, Sworder [24] and Wonham [29] solved this problem for finite T. Sworder used a stochastic maximum principle to obtain his result, and Wonham used dynamic programming. Wonham also solved the infinite time horizon version of this control problem, and derived a set of sufficient conditions for the existence of a unique, finite steady-state solution. Mariton [19] considered a discount cost version of the problem, where a controller that ensures stability in all forms was obtained. A discussion about this result appears in [20]. Mariton and Bertrand [21] also considered an output feedback version of the JLQ problem.

Sworder and Robinson [25] considered problems with u- and x-dependent form transition rates. Such formulations can be used to describe systems where the rates of deterioration of the system components or other abrupt changes are dependent on the level of their load or inputs. Sworder and Robinson [25] obtained the nonlinear partial differential equation related to the optimal solution of the x- and u-dependent problem. Unfortunately, the exact solutions of this class of nonlinear partial differential equations are not known in general. However, approximations can be obtained [25].

Discrete-time versions of the JLQ problem were solved for finite-time horizons in Blair and Sworder [4]. Birdwell *et al.* [3] examined the case where matrix A is not dependent on the form process. Necessary and sufficient conditions were given for the existence of steady-state solutions with finite expected costs for the discrete-time JLQ problem in [8]. In [12], controllability and observability definitions for discrete-time Markovian jump linear systems are described; these conditions can be used to determine the existence of finite steady-state JLQ solutions, similar to the concepts of deterministic controllability and observability in the linear quadratic regulator (LQR) problem. In [7] and [11], the JLQ problem for systems with x- and u-dependent form transition probabilities are considered. Related control problems concerning discrete-time jump linear systems appear in [13] and [14].

There are several definitions of stochastic controllability and observability in the literature for nonjumping systems. For jumping systems, a concept of ϵ-controllability for continuous-time jump linear systems was derived in [22]. This is a local controllability idea, following the approach of [15] and [23]. Sufficient conditions for it are based on a set of coupled differential equations. The problem is that for these controllability conditions (e.g., Remark 3 of [22]), the deterministic controllability of each pair $[A_i, B_i]$ does not ensure the existence of finite steady-state control for jump linear systems. This is demonstrated later, in Example 3 (Section VI). Alternative, global approaches of defining stochastic controllability are given in [6], [9], and [30]. However, these conditions deal with linear systems which are stochastic due to additive random noise terms. In the systems under study here, future values of matrices $A(r(t))$ and $B(r(t))$ are random due to their dependence on the future values of $r(t)$. These are not known at time t. However, form transition probabilities are known. The controllability properties for jump linear systems developed here cannot be obtained by directly extending the results of [6], [9], or [30].

In this paper, we only address the time invariant case as in system (1)–(3), where A, B, Q, R, and λ_{ij}'s are not explicitly dependent on time. Thus, without loss of generality, we let $t_0 = 0$ in (4). And, for notational simplicity, when $r(t) = i$ we will denote $A(r(t))$ and $B(r(t))$ by A_i and B_i, etc. We will let M' denote the transpose of matrix M, $\mu_{\max}(M)$ be the eigenvalue of M with the largest real part, and $\mu_{\min}(M)$ be the eigenvalue of M with the smallest real part.

This paper does the following.

i) Stochastic controllability and stabilizability concepts for continuous time Markovian jump linear systems are developed (Section II).

ii) Necessary and sufficient conditions for stochastic controllability and stabilizability are established by the stochastic Lyapunov function approach as in Kushner [18] (Section III).

iii) The properties of observability and detectability for jump linear system (1)–(3) are investigated by considering its dual system (Section IV).

iv) Necessary and sufficient conditions for the existence of a finite cost solution to the infinite time horizon JLQ problem are derived based on the stochastic stabilizability and observability properties (Section V).

v) Four illustrative examples are given in Section VI.

II. Stochastic Stabilizability and Controllability

Let $x(t, x_0, r_0, u)$ denote the trajectory of the x-process from the initial state $(x(0) = x_0, r(0) = r_0)$, under the action of admissible control $u(t)$. We introduce the following definitions.

Definition 1—Stochastic Stabilizability and Stochastic Controllability: We say that the system (1), (2) is *stochastically stabilizable* if, for all finite $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathbb{S}$, there exists a linear feedback control law $L(r(t))$ that is constant for each value of $r(t) \in \mathbb{S}$:

$$u(t) = -L(r(t))x(t)$$

with $\|L(r(t))\| < \infty$ such that there exists a symmetric positive definite matrix $\tilde{M}$ satisfying

$$\lim_{T\to\infty} E\left\{\int_0^T x'(t, x_0, r_0, u)x(t, x_0, r_0, u)\,dt|x_0, r_0\right\} \le x_0'\tilde{M}x_0. \quad (5)$$

We say that this system is *stochastically controllable* if, for any $x_0 \in \mathbb{R}^n$, $r_0 \in \mathbb{S}$, $\epsilon > 0$ and given finite $T > 0$, there exists $L(r(t))$ as above for $0 \le t \le T$ such that

$$E\{x'(T, x_0, r_0, u)x(T, x_0, r_0, u)|x_0, r_0\} < \epsilon. \quad (6)$$

□

Note that L depends on T and ϵ as well as on x_0 and r_0. Alternatively, in this stabilizability definition we can require the existence of a positive finite constant $\tilde{C}$, with $\tilde{C}\|x_0\|^2$ replacing $x_0'\tilde{M}x_0$ in (5).

Under the above definition, stochastic stabilizability of a system means that there exists a linear feedback control law which drives the x state from any given initial (x_0, r_0) asymptotically to the origin, in the mean square sense. With stochastic controllability, the x state can be driven into an ϵ-neighborhood of the origin in finite time $T > 0$. Except for certain special cases, it is not possible to drive the x state to the origin in the mean square sense in finite time. Using these definitions, stochastic controllability implies stochastic stabilizability, but the inverse is not true.

III. Conditions for Controllability and Stabilizability

In this section we consider necessary and sufficient conditions for the stochastic controllability and stabilizability of (1)–(3). Recall the following terminology for the finite-state Markov chains [10], [17]. A state is transient if a return to it is not guaranteed after leaving; it is recurrent if not transient. A state i is accessible from state j if it is possible to begin in j and arrive in i with some finite time; states i and j are said to communicate if each is accessible from the other. As in [8], a communicating class is a set of states all accessible from each other; these states may be transient or recurrent. A closed communicating class is a communicating class from which exit is not possible. An absorbing state is a single element closed communicating class. A form which is not in any communicating class is said to be noncommunicating. The results developed in this paper will depend upon the above classifications.

Theorem 1: System (1), (2) is stochastically stabilizable if and only if, for each form $i \in \mathbb{S}$, there exists a control law $u(t) = -L_i x(t)$ such that for any given positive-definite symmetric matrix N_i, the (unique) set of symmetric solutions, M_i, of the s coupled equations

$$\left(A_i - B_iL_i - \frac{1}{2}\lambda_i I\right)' M_i + M_i\left(A_i - B_iL_i - \frac{1}{2}\lambda_i I\right) + \sum_{j=1, j\neq i}^{s} \lambda_{ij}M_j = -N_i \quad (7)$$

are positive definite for each $i \in \mathbb{S}$. □

The proof of this theorem, contained in the Appendix, is based upon incorporating the formulas of Wonham [29] into the stochastic Lyapunov function framework of Kushner [18]. The proof is much more complex than the corresponding proof for deterministic linear systems. This is because for each form, the stochastic Lyapunov function for system (1)–(3) must incorporate the dynamics and control laws associated with all accessible forms.

Note that if the system has only a single form, then (7) reduces to the condition for deterministic stabilizability. That is, our stochastic stabilizability definition reduces to the deterministic one for deterministic systems. We will demonstrate later that stabilizability in each form is neither necessary (as shown by Example 1) nor sufficient (as shown by Example 3) for the stochastic stabilizability of system (1)–(3). In applying Theorem 1, we can choose $N_i = I$ for each $i \in \mathbb{S}$, because the condition in Theorem 1 holds for any $\{N_i > 0 : i \in \mathbb{S}\}$.

Note that if a set of symmetric $\{M_i > 0 : i \in \mathbb{S}\}$ is given and we must solve for $\{N_i : i \in \mathbb{S}\}$ via (7), then the positive definiteness of $\{N_i\}$ is only sufficient for the stabilizability of (1)–(3), but not necessary. This can be shown by a simple example. We consider a single form system, with

$$A = \begin{bmatrix} -1 & 4 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

If we choose M in (7) to be the identity matrix I, then using (7) we obtain

$$N = \begin{bmatrix} -2 & 4 \\ 4 & -2 \end{bmatrix}$$

which is not positive definite. However, this system is obviously stabilizable, since both poles of A are negative.

The necessary and sufficient conditions stated in Theorem 1 are not easy to check. The following corollary provides a necessary condition for stochastic stabilizability which is easier to test.

Corollary 1: If (1), (2) is stochastically stabilizable, then in each form i, we can choose L_i such that the matrix $(A_i - B_iL_i - \frac{1}{2}\lambda_i I)$ is stable; that is, all its eigenvalues have negative real parts. □

This corollary is proved in the Appendix. Note that the stability of the matrix $(A_i - B_iL_i - \frac{1}{2}\lambda_i I)$ for each $i \in \mathbb{S}$ is *not* sufficient for the stochastic stabilizability of (1)–(3). This is shown by Example 2 in Section VI.

Next we consider conditions for stochastic controllability of jump linear systems. Before doing this, however, it is useful to reexamine the controllability of linear time-invariant systems. Consider a time-invariant linear system (a single form jump linear system):

$$\dot{x}(t) = Ax(t) + Bu(t). \quad (8)$$

Necessary and sufficient conditions for the pair (A, B) to be controllable are well known in terms of the rank of matrix $[B : AB : \cdots : A^{n-1}B]$ and the ability to arbitrarily assign closed-loop eigenvalues. We introduce here an additional necessary condition and a sufficient condition which, although awkward when considering time-invariant systems, provide an approach for generalization in the jump linear system case.

Lemma 1: The following conditions describe the controllability of the pair (A, B) in (8).

Sufficiency: If, for all choices of scalar γ, there exists a matrix L_s such that the equation

$$(A - BL_s)'M_s + M_s(A - BL_s) = -N \quad (9)$$

has, for each positive definite symmetric matrix N, a unique positive definite symmetric solution M_s with

$$\alpha_s \triangleq \frac{\mu_{\min}(N)}{\mu_{\max}(M_s)} > \gamma, \quad (10)$$

then the pair (A, B) is controllable.

Necessity: If (A, B) is controllable, then for all choices of scalar γ, there exists a matrix L_n such that the equation

$$(A - BL_n)'M_n + M_n(A - BL_n) = -N \quad (11)$$

has, for each positive definite symmetric matrix N, a unique positive definite symmetric solution M_n where

$$\alpha_n \triangleq \frac{\mu_{\max}(N)}{\mu_{\min}(M_n)} > \gamma. \quad (12)$$

Proof: We first prove sufficiency. Suppose that (9) and (10) hold. Let μ be an eigenvalue of $(A - BL_s)$ and let z be the eigenvector associated with μ. From (9), we have (for arbitrarily chosen $N = N' > 0$)

$$\bar{z}'[(A - BL_s)'M_s + M_s(A - BL_s)]z = (\mu + \bar{\mu})\bar{z}'M_sz = -\bar{z}'Nz.$$

Here $\bar{z}'$ is the transpose of the conjugate of z, and $\bar{\mu}$ is the conjugate of μ. Thus,

$$\mu + \bar{\mu} = 2\,\mathrm{Re}\,(\mu) = -\bar{z}'Nz/\bar{z}'M_sz.$$

Taking the maximum of both sides and applying (10),

$$2\,\mathrm{Re}\,[\mu(A - BL_s)] = \mu + \bar{\mu} \le -\frac{\mu_{\min}(N)}{\mu_{\max}(M_s)} = -\alpha_s < -\gamma. \quad (13)$$

Now we prove controllability by contradiction. Assume the pair (A, B) is not controllable; that is, there exists a subspace $\mathbb{V} \subset \mathbb{R}^n$, such that for $x_0 \in \mathbb{V}$, no control $u(t)$ can drive $x(t)$ to the origin at any finite time $t = t_1$. From (13), if we let γ tend to infinity, then we can assign the closed-loop eigenvalues arbitrarily far to the left in the complex plane (by choice of L_s). But this means that $x(t)$ can be driven to the origin within finite time using this L_s, which contradicts the assumption.

Next we prove necessity, also by contradiction. Suppose that (A, B) is controllable. Note that (11) is the standard stabilizability condition. Since (A, B) is controllable, for given symmetric $N > 0$ there exist L_n such that (11) has a unique positive definite solution M_n. Now assume that (12) does not hold; that is, there exists a given number γ_1 such that for any choice of L_n, α_n in (12) is less than or equal to γ_1. For eigenvalue μ and associated eigenvector z of $(A - BL_n)$, we have

$$\bar{z}'[(A - BL_n)'M_n + M_n(A - BL_n)]z = (\mu + \bar{\mu})\bar{z}'M_nz = -\bar{z}'Nz$$

from (11), hence,

$$\mu + \bar{\mu} = 2\,\mathrm{Re}\,(\mu) = -\bar{z}'Nz/\bar{z}'M_nz.$$

Taking the maximum of both sides, our assumption on γ_1 implies that

$$2\,\mathrm{Re}\,[\mu(A-BL_n)] = \mu + \bar{\mu} \geq -\frac{\mu_{\max}(N)}{\mu_{\max}(M_n)} = -\alpha_n > -\gamma_1.$$

This means that at least one of the closed-loop poles cannot be placed to the left of $-\gamma_1$. This contradicts the controllability of (A, B). □

It is well known that $-2\,\mathrm{Re}\,[\mu_{\min}(A-BL)]$ and $-2\,\mathrm{Re}\,[\mu_{\max}(A-BL)]$ are lower and upper bounds on the rate of convergence of $x'(t)x(t)$ toward zero. Applying Lemma 1, we have

$$\alpha_s \geq -2\,\mathrm{Re}\,[\mu_{\max}(A-BL)] \qquad \alpha_n \leq -2\,\mathrm{Re}\,[\mu_{\min}(A-BL)].$$

Thus, α_s and α_n are also lower and upper bounds on the rate of convergence of $x'(t)x(t)$ toward zero, respectively.

Now we return to the question of stochastic controllability for jump linear systems. For jump linear system (1)–(3) with more than one form, we cannot talk about pole assignment in LTI terms, because the system is time varying. However, upper and lower bounds on the convergence rate of $E\{x'(t)x(t)|x_0, r_0\}$ can be used. A condition similar to Lemma 1 characterizes stochastic controllability.

Theorem 2: The stochastic controllability of system (1)–(3) is described by the following conditions.

Sufficiency: If, for all given *finite* constant γ and each choice of positive definite symmetric matrices $\{N_i : i \in \mathbb{S}\}$ we have

$$\alpha_s \triangleq \min_{i \in \mathbb{S}} \left\{ \frac{\mu_{\min}(N_i)}{\mu_{\max}(M_i)} \right\} > \gamma \tag{14}$$

where the matrices $\{M_i > 0 : i \in \mathbb{S}\}$ are obtained from (7) by proper choice of L_i, then system (1)–(3) is stochastically controllable.

Necessity: If system (1)–(3) is stochastically controllable, then for all given *finite* constant γ and each choice of positive definite symmetric matrices $\{N_i : i \in \mathbb{S}\}$ we have

$$\alpha_n \triangleq \min_{i \in \mathbb{S}} \left\{ \frac{\mu_{\max}(N_i)}{\mu_{\min}(M_i)} \right\} > \gamma \tag{15}$$

where the matrices $\{M_i > 0 : i \in \mathbb{S}\}$ are obtained from (7) by proper choice of L_i. □

The proof of Theorem 2 is given in the Appendix.

What this theorem says is that for any positive definite $\{N_i\}$ matrices, there exist control laws $\{L_i : i \in \mathbb{S}\}$ that not only meet the stochastic stabilizability conditions of Theorem 1, but also make the quantity α_s or α_n in (14), (15) greater than any fixed finite positive number γ. The quantity α_s serves as a lower bound on the rate of movement of $E\{x'x|x_0, r_0\}$ toward zero, while α_n is an upper bound.

The necessary and sufficient conditions for stochastic controllability of jump linear systems, that are given in Theorem 2, are conceptually pleasing in that they correspond to stochastic stabilizability plus extra requirements (14) and (15), and they correspond with the "pole placement" conditions in the deterministic case. The conditions of Theorem 2 are unfortunately very difficult to test, however. In particular, establishing that (14) or (15) is true for all finite $\gamma > 0$ is generally difficult since there is no natural way to search for the appropriate L_i values. A more easily tested necessary condition for stochastic controllability is as follows.

Corollary 2: If (1)–(3) is stochastically controllable, then for each form $i \in \mathbb{S}$, the pair (A_i, B_i) is controllable. □

This corollary is proved in the Appendix.

The deterministic controllability for each form is generally not sufficient for (1)–(3) to be stochastically controllable. Example 3 in Section VI demonstrates this fact. The reason is that jumps in the value of $r(t)$ may prevent $E\{x'(t)x(t)|x_0, r_0\}$ from approaching zero, since the application of impulse functions is not allowed here.

There are a number of special types of jump linear systems where stochastic controllability reduces to simpler conditions. They are examined below, because these types of systems have appeared in previous discussions of jump linear systems in the literature.

Corollary 3: For any noncommunicating form i (i.e., noncommunicating transient or absorbing forms), the conditions of Theorem 2 reduce to deterministic controllability of (A_i, B_i). □

This corollary is proved in the Appendix.

Corollary 4: For scalar systems, deterministic controllability in each form ($B_i \neq 0$ for all $i \in \mathbb{S}$) is equivalent to stochastic controllability of the system.

Proof: Note that for scalar A_i and B_i, (7) becomes

$$(2A_i - 2B_iL_i - \lambda_i)M_i + \sum_{j=1, j\neq i}^{s} \lambda_{ij}M_j = -N_j. \tag{16}$$

Since the pair (A_i, B_i) is controllable ($B_i \neq 0$), we can choose L_i such that

$$|2A_i - 2B_iL_i - \lambda_i| > \sum_{j=1, j\neq i}^{s} |\lambda_{ij}| = \sum_{j=1, j\neq i}^{s} \lambda_{ij}.$$

The coefficient matrix of the simultaneous equation (16) is row diagonally dominant and has a unique solution $\{M_i, i \in \mathbb{S}\}$. Also, L_i can be chosen such that $M_i > 0$. In this scalar case, the left-hand side of (14) is

$$\frac{x'N_ix}{x'M_ix} = \frac{N_i}{M_i} = -(2A_i - 2B_iL_i - \lambda_i)\frac{N_i}{N_i + \sum_{j\neq i} \lambda_{ij}M_j}. \tag{17}$$

Since $\Sigma_{j\neq i}\lambda_{ij}M_j$ is finite, the left-hand side of (17) can be made greater than any given finite positive number, by choice of L_i. Note that $\alpha_s = \alpha_n$ in the scalar case. □

In previous literature concerning steady-state JLQ regulators for continuous-time jump linear systems, only scalar examples or examples with all noncommunicating forms were used. For these special cases, deterministic controllability of each pair (A_i, B_i) implies the stochastic controllability of the system; unfortunately, this is not a general property, as shown later in Example 3.

IV. Duality, Observability, and Detectability

We examine the observability and detectability properties of continuous time jump linear systems in this section. Consider the output equation of system (1)–(3):

$$y(t) = C(r(t))x(t) \tag{18}$$

where $y \in \mathbb{R}^p$ is the x-process output. It would be natural to try to apply duality arguments to obtain observability and detectability results from the stochastic controllability and stabilizability results of Sections II and III. But for jump linear systems, this simple approach fails. To see why, consider the observability Grammian of (1), (2), (18):

$$\Gamma(t_0, t_1) \triangleq \int_{t_0}^{t_1} \Phi'(t, t_0)C'(r(t))C(r(t))\Phi(t, t_0)\, dt$$

where $\Phi(t, t_0)$ is the x state transition matrix of (1) with $u(t) = 0$. If we can find a finite time t_1 such that Γ has rank n for any value of $r(t)$ $(t_0 \le t \le t_1)$, then

$$x_0 = \Gamma^{-1}(t_0, t_1) \int_{t_0}^{t_1} \Phi'(t, t_0) C'(r(t)) \cdot \left\{ y(t) - C(r(t)) \int_{t_0}^{t_1} \Phi(t, \tau) B(r(t)) u(\tau) \, d\tau \right\} dt. \quad (19)$$

That is, the system will be observable in the deterministic sense. Note that here the value of x_0 is found at time t_1; at that time, the values of $r(t)$, $t_0 \le t \le t_1$, which are needed to obtain $\Gamma(t_0, t_1)$ of (19), are available.

The above argument is not valid for the controllability problem of (1), (2), (18), however, if we wish to find a control u which drives x state from x_0 (at $t = t_0$) to any given x_1 at a finite time t_1 by

$$u(t) = -B'(r(t)) \Phi'(t_0, t) \Upsilon^{-1}(t_0, t_1)[x_0 - \Phi(t_0, t_1) x_1]. \quad (20)$$

Since the controllability Grammian Υ is

$$\Upsilon(t_0, t_1) \triangleq \int_{t_0}^{t_1} \Phi'(t_0, \tau) B(r(\tau)) B'(r(\tau)) \Phi(t_0, \tau) \, d\tau, \quad (21)$$

then the control law in (20) is not causal. That is, at time t_0, information about the values of $r(t)$ for $t_0 < t \le t_1$ is not available; thus, $\Upsilon(t_0, t_1)$ in (21) cannot be determined.

The dual system of (1), (2), (18) is given by

$$\dot{x}^*(\tau) = A'(r^*(\tau)) x^*(\tau) - C'(r^*(\tau)) u^*(\tau) \quad (1')$$

$$y(\tau) = B'(r^*(\tau)) x^*(\tau). \quad (18')$$

Here $\tau = T - t$. The underlying process $\{r^*(\tau)\}$, $r^*(\tau) = r(T - \tau)$ takes values in the finite set $\mathbb{S}$. It is anticipative under the new running variable τ [1, pp. 172–178], and thus is not random. Under this τ, $\{r^*(\tau)\}$ is also memoryless. However, we have knowledge of the *posterior* transition probabilities

$$\begin{aligned} p_{ij}^* &\triangleq \Pr\{r^*(\tau - \Delta) = j | r^*(\tau) = i\} \\ &= \Pr\{r(T - (\tau - \Delta)) = j | r(T - \tau) = i\} \\ &= \Pr\{r(t + \Delta) = j | r(t) = i\} = p_{ij}. \end{aligned}$$

Thus, by (2), we have

$$p_{ij}^* = \Pr(r^*(\tau - \Delta) = j | r^*(t) = i) = \begin{cases} \lambda_{ij}\Delta + 0(\Delta) & \text{if } i = j \\ 1 + \lambda_{ii}\Delta + 0(\Delta) & \text{if } i \ne j \end{cases} \quad (2')$$

where $\Delta > 0$. This result can be verified by considering the filter problem of a jump linear system with Gaussian input and measurement noises as the regulator problem of the dual system (1′), (2′), (18′), following the approach of [1]. Now we can introduce the observability concept for system (1), (2), (18) as follows.

Definition 2: Jump linear system (1), (2), (18) is said to be observable if there exists a finite time T such that for any (x_0, r_0), the knowledge of input $u(t)$ and output $y(t)$ over $[t_0, T]$ suffices to determine the initial state x_0. □

We have an algebraic test for observability.

Theorem 3: System (1), (2), (18) is observable if and only if, for each form $i \in \mathbb{S}$, the pair (C_i, A_i) is observable. □

Proof: Suppose that $r(t)$ jumps at time instants t_1, $t_2, \cdots, t_\theta$, and

$$\begin{aligned} r(t) &= r_0 = i_0 && \text{for } t_0 \le t < t_1, \\ r(t) &= i_1 && \text{for } t_1 \le t < t_2, \\ &\cdots \\ r(t) &= i_\theta && \text{for } t_\theta \le t \le T. \end{aligned}$$

Since, at time T, the values of $r(t)$ for $t_0 \le t \le T$ are known, we have a deterministic linear time varying system which is piecewise constant in parameters $A(r(t))$ and $C(r(t))$ over $t \in [t_0, T]$. Consider the observability Grammian:

$$\begin{aligned} \Gamma(t_0, T) = &\int_{t_0}^{t_1} \exp[A'_{i_0}(t - t_0)] C'_{i_0} C_{i_0} \exp[A_{i_0}(t - t_0)] \, dt + \cdots \\ &+ \int_{t_\theta}^{T} \exp[A'_{i_0}(t_1 - t_0)] \cdots \exp[A'_{i_0}(t - t_\theta)] C'_{i_\theta} C_{i_\theta} \\ &\cdot \exp[A_{i_\theta}(t - t_\theta)] \cdots \exp[A_{i_0}(t_1 - t_0)] \, dt. \end{aligned} \quad (22)$$

This system is observable if and only if the observability Grammian $\Gamma(t_0, T)$ is positive definite.

We can show that observability of the pair (C_i, A_i), for all $i \in \mathbb{S}$ is both necessary and sufficient for this. Necessity is immediate, since for $r_0 = i \in \mathbb{S}$, it is possible that $r(t)$ stays in i until any finite time T. Lack of observability of (C_i, A_i) implies that $\Gamma(t_0, T)$ is not positive definite. Sufficiency can be seen from writing $\Gamma(t_0, T)$ as

$$\Gamma(t_0, T) = \Gamma_{i_0}(t_0, t_1) + \Gamma_{i_1}(t_1, t_2) + \cdots + \Gamma_{i_\theta}(t_\theta, T)$$

where $\Gamma_{i_0}(t_0, t_1), \cdots, \Gamma_{i_\theta}(t_\theta, T)$ are the corresponding terms on the right-hand side of (22). Thus, $\Gamma_{i_k}(t_k, t_{k+1})$ is the product

$$\begin{aligned} \Gamma_{ik}(t_k, t_{k+1}) = &\exp[A'_{i_0}(t_1 - t_0)] \cdots \exp[A'_{i_{k-1}}(t_{k-1} - t_k)] \tilde{\Gamma}(t_k) \\ &\cdot \exp[A_{i_{k-1}}(t_{k-1} - t_k)] \exp[A_{i_0}(t_1 - t_0)] \end{aligned}$$

where

$$\tilde{\Gamma}(t_k) = \int_{t_k}^{t_{k+1}} \exp[A'_{i_k}(\tau - t_k)] C'_{i_k} C_{i_k} \exp[A'_{i_k}(\tau - t_k)] \, d\tau.$$

The deterministic observability of each pair (C_i, A_i) implies that $\tilde{\Gamma}(t_k)$ is positive definite if $t_k > t_{k-1}$.

We know that the matrix

$$\exp[A_{i_{k-1}}(t_{k-1} - t_k)] \cdots \exp[A_{i_0}(t_1 - t_0)]$$

is nonsingular, so $\Gamma_{i_k}(t_k, t_{k+1}) > 0$; hence, $\Gamma(t_0, T) > 0$ and system (1), (2), (18) is observable. □

Based on the above discussion, we can treat the detectability (in the sense of [28]) of (1), (2), (18) like that for a deterministic, time varying system. Assuming that $\{u(t), t \in [t_0, T]\}$ is known, consider the asymptotic x state estimator (observer)

$$d\hat{x}(t)/dt = A(r(t))\hat{x}(t) + B(r(t))u(t) + H(r(t))[y(t) - C(r(t))\hat{x}(t)].$$

Then we have the following definition.

Definition 3: We say that system (1), (2), (18) is detectable if there exists $\{H_i : i \in \mathbb{S}\}$ for the above estimator such that the estimate error

$$\tilde{x}(t) \triangleq x(t) - \hat{x}(t)$$

is globally, asymptotically Lyapunov stable. □

For detectability, we have the following immediate conditions.

Theorem 4: System (1), (2), (18) is detectable if and only if there exist H_i for each form $i \in \mathbb{S}$, such that for any initial condition (x_0, r_0) and on any form sample path $\{i_0, i_1, \cdots, i_\theta\}$, we have

$$\lim_{T\to\infty} |\exp[(A'_{i_\theta} - C'_{i_\theta}H'_{i_\theta})(T - t_\theta)] \cdots \exp[(A'_{i_0} - C'_{i_0}H'_{i_0})(t_1 - t_0)]x_0| = 0. \quad \square$$

Corollary 5:

i) A necessary condition for (1), (2), (18) to be detectable is that for each form $i \in \mathbb{S}$, the pair (C_i, A_i) is detectable.

ii) A sufficient condition for (1), (2), (18) to be detectable is that for each form $i \in \mathbb{S}$, there exists H_i such that

$$\|\exp((A'_i - C'_iH'_i)\| < 1.$$

iii) If there is no communicating form in $\mathbb{S}$, then (1), (2), (18) is detectable if and only if for each $i \in \mathbb{S}$, the pair (C_i, A_i) is detectable. $\square$

It is easy to see that observability implies detectability of system (1), (2), (18), since the observability of each pair (C_i, A_i) ensures the sufficient condition (ii) of Theorem 4.

V. The Infinite Time Markov JLQ Optimal Control Problem

The definitions of Sections II–IV have been constructed so that they play the same role for the infinite time horizon jump linear quadratic problem that the deterministic properties play for the linear quadratic regulator problem. In particular, they determine the existence and uniqueness of constant (in time for each form value) control laws that stabilize the system with finite expected cost.

First recall the results of finite time JLQ problem [24], [29]. For system (1)–(3) with cost criterion (4), the optimal control law is

$$u(t, i) = -L_i(t)x(t)$$

where

$$L_i(t) = R_i^{-1}B'_iK_i(t) \qquad \text{for } r(t) = i \tag{23}$$

with cost-to-go

$$G_i(x, t, T) = J(t_0, x_0, r_0, T) - J(t_0, x_0, r_0, t) = x'(t)K_i(t)x(t) \tag{24}$$

where the $K_i(t)$ are the unique, positive semidefinite solutions of the following set of coupled matrix Riccati equations:

$$\dot{K}_i(t) + [A_i - B_iL_i(t)]'K_i(t) + K_i(t)[A_i - B_iL_i(t)] - \lambda_iK_i(t) + \sum_{j=1, j\neq i}^{s} \lambda_{ij}K_j(t) + Q_i + L'_iR_iL_i = 0 \tag{25}$$

with $L_i(t)$ defined in (23) and terminal conditions

$$K_i(T) = 0 \qquad i \in \mathbb{S}, \qquad t_0 \le t \le T. \tag{26}$$

For the case of $T \to \infty$, we have the following result.

Theorem 5: For JLQ problem (1)–(4) with $T \to \infty$, assume that for $C'_iC_i = Q_i$, system (1), (2), (18) is observable. Then if and only if the system is stochastically stabilizable, the solution $K_i(t)$ of (25) for each $i \in \mathbb{S}$ is finite and

$$K_i = \lim_{T-t\to\infty} K_i(t)$$

exists, where $K_i(i \in \mathbb{S})$ is the set of unique, positive solutions of the coupled algebraic matrix Riccati equations

$$A'_iK_i + K_iA_i - K_iB_iR_i^{-1}B'_iK_i - \lambda_iK_i + \sum_{j=1, j\neq i}^{s} \lambda_{ij}K_j + Q_i = 0. \tag{27}$$

The optimal steady state control is

$$u(t) = -L_ix(t) = -R_i^{-1}B'_iK_ix(t)$$

and the controlled system is stable in the sense that

$$E\{x'(t)x(t)|x_0, r_0\} \to 0 \qquad \text{as } t \to \infty.$$

Proof Sketch: With the stochastic stabilizability assumption, there exist $\tilde{L}_i$ (not necessarily optimal) for each $i \in \mathbb{S}$ such that for any given $N_i > 0$ $(i \in \mathbb{S})$, there exists a unique symmetric set $\{M_i > 0 : i \in \mathbb{S}\}$ which solves (7). Using the observability assumption, we show that for such a choice of $\tilde{L}_i$, $(Q_i + \tilde{L}'_iR_i\tilde{L}_i) > 0$ for each $i \in \mathbb{S}$. By Theorem 3, each pair $((Q_i)^{1/2}, A_i)$ is observable; thus, the pair $((Q_i)^{1/2}, A_i - \frac{1}{2}\lambda_iI)$ is observable, which implies the observability of the pair $((Q_i + \tilde{L}'_iR_i\tilde{L}_i)^{1/2}, A_i - B_i\tilde{L}_i - \frac{1}{2}\lambda_iI)$ (by [28, Lemma 4.1]). Therefore, for any $t > 0$,

$$D_i = \int_0^t \exp\left[\left(A_i - B_i\tilde{L}_i - \frac{1}{2}\lambda_iI\right)'\tau\right](Q_i + \tilde{L}'_iR_i\tilde{L}_i) \cdot \exp\left[\left(A_i - B_i\tilde{L}_i - \frac{1}{2}\lambda_iI\right)\tau\right]d\tau$$

which is the observability Grammian of the pair $((Q_i + \tilde{L}'_iR_i\tilde{L}_i)^{1/2}, A_i - B_i\tilde{L}_i - \frac{1}{2}\lambda_iI)$. It is bounded (by [28]) and positive definite. Since $D_i > 0$ and the matrix $(A_i - B_i\tilde{L}_i - \frac{1}{2}\lambda_iI)$ is stable (from Corollary 1), we have $(Q_i + \tilde{L}'_iR_i\tilde{L}_i) > 0$. With this fact in hand, we can show that the JLQ cost in (4) for this problem is a Lyapunov function for the controlled system. Specifically, with

$$N_i \triangleq Q_i + \tilde{L}'_iR_i\tilde{L}_i > 0$$

then (4) becomes

$$\begin{aligned}
&E\left\{\int_{t_0}^{\infty}[x'(t)Q(r(t))x(t) + u'(t)R(r(t))u(t)]\,dt\right\}\\
&= E\left\{\int_{t_0}^{\infty}[x'(t)[Q(r(t)) + \tilde{L}'(t)R(r(t))\tilde{L}(r(t))]x(t)\,dt\right\}\\
&= E\left\{\int_{t_0}^{\infty}[x'(t)N(r(t))x(t)\,dt\right\}\\
&= E\left\{\int_{t_0}^{\infty} -\tilde{A}V(x(t), r(t))\,dt\right\}\\
&= E\left\{\int_{t_0}^{\infty} -\operatorname*{Lim}_{\Delta\to 0}\frac{1}{\Delta}[E\{V(x(t+\Delta), r(t+\Delta)|x(t), r(t)\} - V(x(t), r(t)]\,dt\right\}\\
&= E\left\{-\operatorname*{Lim}_{\Delta\to 0}\sum_{k=0}^{\infty}\Delta\frac{1}{\Delta}[E\{V(x(t_0+(k+1)\Delta), r(t_0+(k+1)\Delta))|x(t_0+k\Delta), r(t_0+\Delta)\} - V(x(t_0+\Delta), r(t_0+\Delta))]\right\}\\
&= -E\{V(x(t=\infty), r(t=\infty))|x_0, r_0\} + V(x_0, r_0)
\end{aligned} \tag{28}$$

where $\tilde{A}V(x(t), r(t))$ is defined in (A.3) of the Appendix.

The first term on the right-hand side of (28) is zero from (A.6). Without loss of generality, let $r(t_0) = r_0 = i$; note that the left-hand side of (28) is the cost-to-go from $(x_0, r_0 = i)$ and the right-hand side equals the Lyapunov function

$$V(x_0, r_0 = i) = x_0' M_i x_0.$$

This finite quantity (resulting from a not-necessarily optimal feedback control $\{\bar{L}_i : i \in \mathbb{S}\}$) serves as an upper bound for the optimal cost-to-go $x'(t)K_i(t)x(t)$ (resulting from the optimal choice of $\{L_i : i \in \mathbb{S}\}$). Each $K_i(t)$ [obtained from (25)] is monotonically nondecreasing as T increases. Therefore, each $K_i(t)$ converges to constant matrix K_i as $(T - t) \to \infty$. Thus, (25) becomes (27). Since $x_0' K_i x_0 < \infty$ as $T \to \infty$, the positive definiteness of each K_i and the observability of the system ensure that for this closed-loop system with the optimal feedback control, we have $E\{x'(t)x(t)|x_0, r_0\} \to 0$ as $t \to \infty$.

The above proves sufficiency. With the observability assumption, the proof that stochastic stabilizability is necessary for the existence of a finite solution is immediate. Suppose the system is not stochastically stabilizable; that is, no feedback control law can result in a finite cost. Then the fact that the optimal control results in a finite cost is a contradiction. □

The observability requirement in Theorem 5 is not necessary for the stability of the controlled system. It may be that replacing observability with detectability in Theorem 5 will lead to a complete set of necessary and sufficient conditions for finite cost and stability of controlled system. However, since the positive definiteness of $(Q_i + L_i' R_i L_i)$ cannot be established by detectability of (1), (2), (18) alone, it may be necessary to modify the stochastic stabilizability condition to allow positive semidefinite N_i to be chosen, under some type of detectability assumption.

Note that the choice of feedback laws $\{L_j : j \in \mathbb{S}\}$ could be viewed as a stabilization problem for (1)–(3). That is, we simultaneously select, for each form $j \in \mathbb{S}$, a closed-loop dynamic matrix $\bar{A}_j$ such that the equations

$$\dot{x} = \bar{A}_j x$$

have desired stability properties. These choices are made from the sets $\{A_j - B_j L_j : L_j \in \mathbb{R}^{m \times n}\}$. The desired stability properties can be obtained from the results of Theorem 1 and Corollary 1 by setting $u \equiv 0$.

VI. Examples

Wonham [29] gave an example to show that the optimal JLQ controller does not necessarily stabilize all $(A_i - B_i L_i)$. This is because the stability of $(A_i - B_i L_i)$ for each $i \in \mathbb{S}$ only means stability on constant sample paths of $r(t)$. The JLQ controller, which obtains mean square stability, does not guarantee stability on all form sample paths, and thus does not guarantee the stability of all $(A_i - B_i L_i)$.

In this section, four examples are developed to illustrate the results established in Sections I–V. Example 1 is used to illustrate two points. First, it shows that stochastic stabilizability of system (1)–(3) does not imply that each form is deterministically stabilizable. Second, it demonstrates that a sufficient condition of Wonham [29, (6.12)] is not necessary for finite cost solution to the infinite time JLQ problem.

Example 1: We consider a two-form system, with

$$\Lambda = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \qquad A_1 = \frac{1}{3} \qquad A_2 = -\frac{4}{3} \qquad B_1 = B_2 = 0.$$

This system is not stabilizable in form 1 since $A_1 > 0$ and $B_1 = 0$. To test stochastic stabilizability by Theorem 1, we solve (7) for this system as follows:

$$\begin{cases} -\frac{1}{3}M_1 + M_2 + 1 = 0 \\ -\frac{11}{3}M_2 + M_1 + 1 = 0. \end{cases}$$

Since the solution $M_1 = 21 > 0$, $M_2 = 6 > 0$, this system is stochastically stabilizable. Thus, the stabilizability of (A_i, B_i) $\forall i \in \mathbb{S}$ is not necessary for the system to be stochastically stabilizable. Next, we know by Theorem 5, there exists a finite cost solution for the infinite time JLQ problem of this system. But consider the condition

$$\|\lambda_i' \int_0^\infty \exp\left[\left(A_i - B_i L_i - \frac{1}{2}\lambda_i I\right)' t\right] \cdot \exp\left[\left(A_i - B_i L_i - \frac{1}{2}\lambda_i I\right) t\right] dt \| < 1 \qquad (29)$$

which [29, (6.12)] was given by Wonham as part of a set of sufficient conditions for the existence of a finite cost solution to the infinite time JLQ problem. It is easy to see that for $i = 1$, the left-hand side of (29) equals 3; thus, (29) is not satisfied. Thus, conditions (i) and (ii) in Theorem 6.1 of [29] are sufficient but not necessary for the stochastic stabilizability of the system.

The next example demonstrates that the necessary condition of Corollary 1, the stability of $(A_i - B_i L_i - 1/2\lambda_i)$, $\forall i \in \mathbb{S}$, is *not* sufficient for stochastic stabilizability.

Example 2: Consider a two-form system, with

$$\Lambda = \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} \qquad A_1 = 1 \qquad A_2 = \frac{1}{2} \qquad B_1 = B_2 = 0.$$

To test for stochastic stabilizability of this system, we apply Theorem 1. For any choice of $\{L_i : i = 1, 2\}$, we obtain from (7)

$$\begin{cases} 2M_1 - 3M_1 + 3M_2 = -1 \\ M_2 - 2M_2 + 2M_1 = -1. \end{cases}$$

The only solution is $M_1 = -4/5$, $M_2 = -3/5$; they are not positive definite. Thus, by Theorem 1, this system is not stabilizable. But we have $(A_1 - B_1 L_1 - \frac{1}{2}\lambda_1) = -1/2$ and $(A_2 - B_2 L_2 - \frac{1}{2}\lambda_2) = -1/2$. That is, they are both stable. Thus, we see that this "decoupled" (in terms of parameters in each form) necessary condition of Corollary 1 is not sufficient for stochastic stabilizability.

The next example illustrates that controllability of each pair (A_i, B_i) $\forall i \in \mathbb{S}$ is not sufficient for stochastic stabilizability of system (1)–(3); consequently, it is not sufficient for the stochastic controllability of system (1)–(3).

Example 3: Consider a two-form system with

$$\Lambda = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 1.5 & 1 \\ 0 & 0.5 \end{bmatrix} \qquad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0.5 & 0 \\ 1 & 1.5 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Note that each pair (A_i, B_i) $i \in \{1, 2\}$ is controllable. Taking $C_1 = C_2 = I$, by Theorem 3, this system is observable. Let $R_1 = R_2 = 1$, $Q_1 = C_1' C_1 = I$, $Q_2 = C_2' C_2 = I$, and consider the infinite time JLQ problem. The two sets of real symmetric solutions of (27) are

$$K_1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \qquad K_2 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

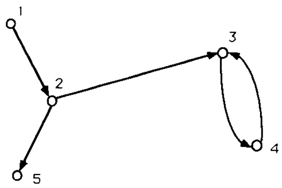

Fig. 1. Form structure of Example 4.

and

$$K_1 = \begin{bmatrix} 6.729043 & -3.834479 \\ -3.834479 & 0.245122 \end{bmatrix}$$

$$K_2 = \begin{bmatrix} 0.245122 & -3.834479 \\ -3.834479 & 6.729043 \end{bmatrix}.$$

In both cases, K_1 and K_2 are not positive definite. Thus, by Theorem 5, this system is not stochastically stabilizable.

This example is interesting because it illustrates a counterintuitive point. The freedom of assignment of closed-loop poles in each form does not ensure that the effects of the underlying jump process can be dominated. Examples 1 and 3 demonstrate that the controllability or stabilizability of each pair (A_i, B_i) is neither necessary nor sufficient for the stochastic stabilizability of the system (1)–(3).

In the next example we demonstrate how to use Theorem 5 to obtain the steady-state optimal controller for a five-form system.

Example 4: Consider the system with transition rate matrix

$$\Lambda = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0.5 & 0 & 0.5 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with the form structure shown in Fig. 1. Here forms 1 and 2 are noncommunicating transient forms, 5 is an absorbing form, and $\{3, 4\}$ is a closed communicating class. Let

$$A_1 = \begin{bmatrix} 1.5 & 1 \\ 0 & -1.5 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1.5 & 0 \\ 1 & 1.5 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad B_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \quad B_5 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with $R_i = 1$ and $Q_i = I$ in (4) for $i = 1, 2, 3, 4, 5$. Consider the infinite time horizon JLQ problem. First note that with $C_i = I = Q_i^{1/2}$, the pairs (C_i, A_i) are observable. So the system is observable by Theorem 3. For the noncommunicating transient forms $\{1, 2\}$ and absorbing form $\{5\}$, the pairs (A_i, B_i) are all controllable. Thus, by Corollary 3, for these forms, the conditions of Theorem 2 (stochastic controllability) can be satisfied by proper choice of L_i. Thus, the conditions for Theorem 1 (stochastic stabilizability) are satisfied for forms $\{1, 2, 5\}$. Since only forms 3 and 4 are communicating forms, we check (7) for them. That is,

$$\left(A_3 - \frac{1}{2}\lambda_3 I - B_3 L_3\right)' M_3 + M_3 \left(A_3 - \frac{1}{2}\lambda_3 I - B_3 L_3\right) + \lambda_3 M_4 = -N_3$$

$$\left(A_4 - \frac{1}{2}\lambda_4 I - B_4 L_4\right)' M_4 + M_4 \left(A_4 - \frac{1}{2}\lambda_4 I - B_4 L_4\right) + \lambda_4 M_3 = -N_4.$$

With the choices of $L_3 = [1 \ \ 2]$ for form 3 and $L_4 = [2 \ \ 1]$ for form 4, for

$$N_3 = \begin{bmatrix} 4/7 & -1/7 \\ -1/7 & 2/7 \end{bmatrix} \qquad N_4 = \begin{bmatrix} 2/7 & -1/7 \\ -1/7 & 4/7 \end{bmatrix},$$

we obtain the positive definite solutions of (7)

$$M_3 = \begin{bmatrix} 5/7 & 1/7 \\ 1/7 & 3/7 \end{bmatrix} \qquad M_4 = \begin{bmatrix} 5/7 & 1/7 \\ 1/7 & 3/7 \end{bmatrix}.$$

Thus, the conditions of Theorem 1 are satisfied for forms 3 and 4. Since the conditions of Theorem 1 are satisfied for all five forms, this system is stochastically stabilizable. Therefore, by Theorem 5, the steady-state solution for this problem exists and can be found by solving (27). We can obtain the solution as follows. First find K_5, where (27) is a deterministic algebraic Riccati equation. Then find K_3 and K_4, by solving two coupled algebraic Riccati equations. With K_3 and K_5 known, K_2 can be found from (27). Finally, with K_2 known, K_1 can be obtained from (27). The result is the following:

$$K_5 = \begin{bmatrix} 2.02486 & 0.41421 \\ 0.41421 & 2.04244 \end{bmatrix}$$

$$K_4 = K_3 = \begin{bmatrix} 1.73205 & 1 \\ 1 & 1.73205 \end{bmatrix}$$

$$K_2 = \begin{bmatrix} 1.138647 & 1.486321 \\ 1.486321 & 2.47161 \end{bmatrix}$$

$$K_1 = \begin{bmatrix} 27.63533 & 7.576919 \\ 7.576919 & 2.843495 \end{bmatrix}$$

with steady-state feedback gains:

$$L_5 = [7.5765919 \quad 2.843495] \qquad L_4 = [1.73205 \quad 1]$$

$$L_3 = [1 \quad 1.73205]$$

$$L_2 = [0.35857 \quad 3.28044] \qquad L_1 = [3.451813 \quad 0.529775].$$

VII. Summary

Concepts of stochastic controllability and stochastic stabilizability are developed here for continuous-time Markov jump linear systems. Corresponding necessary and sufficient conditions are given. The observability and detectability concepts used are deterministic, however. Together, these properties are related to the solution of the continuous-time jump linear quadratic control problem with infinite time horizon. Based on these properties, necessary and sufficient conditions for the existence of a finite cost steady-state solution which stabilizes the controlled system are obtained.

Appendix

Proof of Theorem 1: Following [18], we first set up the stochastic Lyapunov function for the closed-loop system. Consider feedback control $u(t) = -L_i x(t)$; then (1) becomes

$$dx(t)/dt = (A_i - B_i L_i)x(t) \qquad r(t) = i \in \mathbb{S}. \tag{A.1}$$

Take the stochastic Lyapunov function to be

$$V(x(t), r(t) = i) = V(x, i) = x'(t)M_i x(t). \tag{A.2}$$

Note that M_i is constant for each i. For this choice of $V(x, r)$, we have $V(0, r) = 0$ and $V(x, r) \to \infty$ only when $\|x\| \to \infty$.

Consider the weak infinitesimal operator $\tilde{A}$ of the joint process $\{(r(t), x(t), t \in [t_0, T]\}$, which is the natural stochastic analog of the deterministic derivative [18]. Here the domain of $\tilde{A}$ is the function space $[t_0, T] \times \mathbb{S} \times \mathbb{R}^n$. Note that because $\mathbb{S}$ is a finite set, $\tilde{A}$ is also the generator of $\{r(t), x(t)\}$, so from [29, equation (2.26)], we have

$$\begin{aligned}\tilde{A}V(x, i) &\triangleq \operatorname*{Lim}_{\Delta \to 0} \frac{1}{\Delta}[E\{V(x(t+\Delta), r(t+\Delta)|x(t), r(t) = i\} \\ &\quad - V(x(t), r(t) = i)] \\ &= x'\left[(A_i - B_i L_i)'M_i + M_i(A_i - B_i L_i) \right. \\ &\quad \left. + \sum_{j=1}^{s} \lambda_{ij} M_j\right] x \\ &= x'\left[\left(A_i - B_i L_i - \frac{1}{2}\lambda_i I\right)' M_i + M_i \right. \\ &\quad \left. \cdot \left(A_i - B_i L_i - \frac{1}{2}\lambda_i I\right) + \sum_{j=1, j\neq i}^{s} \lambda_{ij} M_j\right] x.\end{aligned} \tag{A.3}$$

With stochastic Lyapunov function of (A.2) and its weak infinitesimal operator of (A.3), we can prove sufficiency: by (7), we have $\tilde{A}V(x, i) = -x'N_i x$. Thus,

$$\frac{\tilde{A}V(x, i)}{V(x, i)} = -\frac{x'N_i x}{x'M_i x} = -\beta_i(x) \qquad \text{for } x \neq 0. \tag{A.4}$$

Since $M_i > 0$ and $N_i > 0$, $\beta_i(x)$ is a positive number and we have

$$-\beta_i(x) \le -\alpha \triangleq -\min_{i \in \mathbb{S}} \frac{\mu_{\min}(N_i)}{\mu_{\max}(M_i)}. \tag{A.5}$$

Clearly $\alpha > 0$. So

$$\tilde{A}V(x, i) \le -\alpha V(x, i).$$

Then by Dynkin's formula and the Gronwell–Bellman lemma, we have for all $i \in \mathbb{S}$

$$E\{V(x(t), i)\} \le \exp(-\alpha t)V(x_0, i).$$

Thus,

$$\begin{aligned}E\{V(x(t), i)|x_0, r_0 = i\} &= E\{x'(t)M_i x(t)|x_0, r_0 = i\} \\ &\le \exp(-\alpha t)x_0' M_i x_0.\end{aligned} \tag{A.6}$$

(Similar arguments for deterministic system Lyapunov functions can be found in [5].) Thus, we have

$$\begin{aligned}E\left\{\int_0^T x'(t)M_i x(t)\,dt|x_0, r_0 = i\right\} &\le \left(\int_0^T \exp(-\alpha t)\,dt\right) x_0' M_i x_0 \\ &= -\frac{1}{\alpha}[\exp(-\alpha T) - 1]x_0' M_i x_0.\end{aligned}$$

Taking limit as $T \to \infty$, we have

$$\operatorname*{Lim}_{T \to \infty} E\left\{\int_0^T x'(t)M_i x(t)\,dt|x_0, r_0 = i\right\} \le \frac{1}{\alpha}x_0' M_i x_0.$$

Since $M_i > 0$ for each $i \in \mathbb{S}$, this means

$$\operatorname*{Lim}_{T \to \infty} E\left\{\int_0^T x'(t)x(t)\,dt|x_0, r_0 = i\right\} \le x_0'\tilde{M}x_0 \tag{A.7}$$

where

$$\tilde{M} = \operatorname*{Max}_{i \in \mathbb{S}} \frac{M_i}{\alpha\|M_i\|}$$

which proves sufficiency.

For necessity, we must show that if system (1)–(3) is stochastically stabilizable, then the conditions of Theorem 1 hold. That is, for any choice of $\{N_i > 0 : i \in \mathbb{S}\}$, there exists a unique set of symmetric solutions, $\{M_i > 0 : i \in \mathbb{S}\}$, of (7). We assume that the system (1), (2) is stochastically stabilizable. That is, for appropriately chosen $u(t) = -L_i x(t)$ in (A.1), we have

$$\operatorname*{Lim}_{T \to \infty} E\left\{\int_0^T x'(t)x(t)\,dt|x_0, r_0 = i\right\} \le x_0'\tilde{M}x_0. \tag{A.8}$$

Consider the following function:

$$x'(t)M(T-t, r(t))x(t) \triangleq E\left\{\int_t^T x'(\tau)N(r(\tau))x(\tau)\,d\tau|x(t), r(t)\right\}. \tag{A.9}$$

Assume $x(t) \neq 0$. Since $N(r(t)) > 0$, then as T increases, either $x'(t)M(T-t, r(t))x(t)$ is monotonically increasing or else it increases monotonically until

$$E\{x'(\tau)N(r(\tau))x(\tau)|x(t), r(t)\} = 0$$

for all $\tau \ge \tau_1 \ge t$. From (A.8), we know $x'(t)M(T-t, r(t))x(t)$ is bounded above. Thus, the following limit exists:

$$\begin{aligned}x'(t)M_i x(t) &\triangleq \operatorname*{Lim}_{T \to \infty} x'(t)M(T-t, r(t) = i)x(t) \\ &= \operatorname*{Lim}_{T \to \infty} E\left\{\int_t^T x'(\tau)N(r(\tau))x(\tau)\,d\tau|x(t), r(t)\right\}.\end{aligned} \tag{A.10}$$

Since this is valid for any $x(t)$, we have

$$M_i \triangleq \operatorname*{Lim}_{T \to \infty} M(T-t, r(t) = i). \tag{A.11}$$

From (A.10), we see that M_i is symmetric and positive definite because N_i is symmetric and positive definite. Now consider

$$\begin{aligned}E\{x'(t)M(T-t, r(t)=i)x(t) - x'(t+\Delta)\\ \cdot M(T-t-\Delta, r(t+\Delta))x(t+\Delta)|x(t), r(t)=i\}\\ = E\left\{E\left\{\int_t^{t+\Delta} x'(\tau)N(r(\tau))x(\tau)\,d\tau | x(t), r(t)=i\right\}\right.\\ \left.|x(t), r(t)=i\right\}.\end{aligned} \tag{A.12}$$

By ignoring the higher order terms $0(\Delta)$, (A.12) can be expressed as

$$\begin{aligned}&-x'(t)[(A_i-B_iL_i)'M(T-t, r(t)=i)\Delta - M(T-t, r(t)=i)\\ &\cdot(A_i-B_iL_i)\Delta + \Delta\sum_{j=1}^{s}\lambda_{ij}M(T-t-\Delta, r(t+\Delta)=j)\\ &+\Delta\dot{M}(T-t, r(t)=i)]x(t)\\ &= E\left\{\int_t^{t+\Delta} x'(\tau)N(r(\tau))x(\tau)\,d\tau | x(t), r(t)=i\right\}.\end{aligned} \tag{A.13}$$

Dividing both sides of (A.13) with Δ, then taking limit as $\Delta \to 0$, we have

$$\begin{aligned}&-x'(t)[(A_i-B_iL_i)'M(T-t, r(t)=i) + M(T-t, r(t)=i)\\ &\cdot(A_i-B_iL_i) + \sum_{j=1}^{s}\lambda_{ij}M(T-t, r(t)=j)\\ &+\dot{M}(T-t, r(t)=i)]x(t)\\ &= E\left\{\lim_{\Delta\to 0}\int_t^{t+\Delta} x'(\tau)N(r(\tau))x(\tau)\,d\tau | x(t), r(t)=i\right\}\\ &= x'(t)N_ix(t).\end{aligned}$$

Note that it must be valid for any $x(t)$. Thus, we have

$$\begin{aligned}&-(A_i-B_iL_i)'M(T-t, r(t)=i) - M(T-t, r(t)=i)\\ &\cdot(A_i-B_iL_i) - \dot{M}(T-t, r(t)=i)\\ &-\sum_{j=1}^{s}\lambda_{ij}M(T-t, r(t)=j) = N_i.\end{aligned} \tag{A.14}$$

From (A.11), $M(T-t, r(t))$ is not time varying as $T-t$ tends to infinity; it converges to M_i. So taking the limit as $T-t\to\infty$ on both sides of (A.14), we have (7); $\{M_i(i\in\mathbb{S})\}$ is the unique set of symmetric, positive definite solutions. This proves necessity. □

Proof of Corollary 1: In the proof of the necessity of Theorem 1, from (A.14), we have

$$\begin{aligned}\frac{d}{dt}M(T-t, r(t)=i) &= -\mathcal{A}_i'M(T-t, r(t)=i)\\ &-M(T-t, r(t)=i)\mathcal{A}_i - \tilde{N}(T-t, r(t))\end{aligned} \tag{A.15}$$

where

$$\mathcal{A}_i = A_i - B_iL_i - \frac{1}{2}\lambda_iI \qquad \tilde{N}(T-t, r(t)) = N_i + \sum_{j=1, j\neq i}^{s}\lambda_{ij}M(T-t, r(t)=j),$$

that is,

$$\begin{aligned}\frac{d}{d(T-t)}M(T-t, r(t)=i) &= \mathcal{A}_i'M(T-t, r(t)=i)\\ &+M(T-t, r(t)=i)\mathcal{A}_i - \tilde{N}(T-t, r(t)).\end{aligned} \tag{A.16}$$

The solution of (A.16) can be expressed as

$$\begin{aligned}M(T-t, r(t)=i) &= \exp[\mathcal{A}_i'(T-t)]M(t, r(t)=i)\\ &\cdot\exp[\mathcal{A}_i'(T-t)] + \int_t^{T-t}\\ &\cdot\exp[\mathcal{A}_i'(T-t-\tau)]\tilde{N}(T-t-\tau, r(t-\tau))\\ &\cdot\exp[\mathcal{A}_i'(T-t-\tau)]\,d\tau.\end{aligned} \tag{A.17}$$

From (A.11), we know $M(T-t, r(t)=i)$ has a limit M_i when T tends to infinity for any initial value of $M(t, r(t)=i)$. So for $M(t, r(t)=i) = I$ in (A.17), taking the limit over both sides of (A.17), we obtain

$$\begin{aligned}M_i = &\lim_{T-t\to\infty}\{\exp[(T-t)\mathcal{A}_i']\exp[(T-t)\mathcal{A}_i]\}\\ &+\lim_{T-t\to\infty}\left\{\int_t^{T-t}\exp[\mathcal{A}_i'(T-t-\tau)]\right.\\ &\cdot\tilde{N}(T-t-\tau, r(t-\tau))\\ &\left.\cdot\exp[\mathcal{A}_i'(T-t-\tau)]\,d\tau\right\}\end{aligned} \tag{A.18}$$

and for $M(t, r(t)=i) = 0$, we have

$$\begin{aligned}M_i = \lim_{T-t\to\infty}&\left\{\int_t^{T-t}\exp[\mathcal{A}_i'(T-t-\tau)]\right.\\ &\left.\cdot\tilde{N}(T-t-\tau, r(t-\tau))\exp[\mathcal{A}_i'(T-t-\tau)]\,d\tau\right\}.\end{aligned}$$

Thus, the first term on the right in (A.18) equals zero. That is, for any vector b in $\mathbb{R}^n$,

$$\lim_{T-t\to\infty}\{b'\exp[(T-t)\mathcal{A}_i']\exp[(T-t)\mathcal{A}_i]b\} = 0$$

which implies that

$$\lim_{T-t\to\infty}\{\exp[(T-t)\mathcal{A}_i]b\} = 0. \tag{A.19}$$

Note that (A.19) states that $\mathcal{A}_i = (A_i - B_iL_i - 1/2\lambda_iI)$ is stable. This proves the corollary. □

Proof of Theorem 2: As in the proof of Theorem 1, for the closed-loop system with feedback control $u(t) = -L_iX(t)$, the stochastic Lyapunov function is

$$V(x(t), r(t)=i) = V_i(x) = x'(t)M_ix(t).$$

We first prove sufficiency. Since (1)–(3) is stochastically stabilizable, from (A.4)–(A.6) we have

$$E\{x'(T)M_ix(T)|x_0, r_0=i\} \le \exp[-\alpha_s(T-t_0)]x_0'M_ix_0.$$

Here α_s is defined in (14) and M_i is given by (7). Note that $M_i > 0$, so we can write it as $M_i = H_i'H_i$, where H_i is nonsingular. Let $z = H_ix$. Then

$$E\{z'(T)z(T)|z_0, r_0=i\} \le \exp[(-\alpha_s)(T-t_0)]z_0'z_0 \tag{A.20}$$

where z_0 is an arbitrary initial state. Thus, it is only a matter of notation to replace z with x:

$$E\{z'(T)x(T)|x_0, r_0 = i\} \leq \exp[(-\alpha_s)(T - t_0)]x_0'x_0.$$

For any choice of $\epsilon > 0$ in the definition of controllability, γ can be chosen large enough such that

$$\gamma \geq |\ln(x_0'x_0) - \ln(\epsilon)|/(T - t_0).$$

This control law $\{L_i : i \in \mathbb{S}\}$ drives $E\{x'(T)x(T)|x_0, r_0 = i\} < \epsilon$. Thus, the system is stochastically controllable.

Next we prove necessity. Suppose the system is stochastically controllable. Since stochastic controllability implies stochastic stabilizability, then for any given $\{N_i > 0 : i \in \mathbb{S}\}$, there exists $\{L_i : i \in \mathbb{S}\}$, such that the solution $\{M_i > 0 : i \in \mathbb{S}\}$ satisfies (7). We prove condition (15) by contradiction. Assume that for any choice $\{L_i : i \in \mathbb{S}\}$, for any given $\{N_i > 0 : i \in \mathbb{S}\}$ and some obtained M_i, we have

$$\alpha_n = \max_{i \in \mathbb{S}} \left\{ \frac{\mu_{\max}(N_i)}{\mu_{\min}(M_i)} \right\} \leq \gamma$$

for some specified finite $\gamma > 0$. Then similar to (A.4)–(A.6), we have

$$\frac{\tilde{A}V_i}{V_i} = -\frac{x'N_i x}{x'M_i x} \geq -\alpha_n \qquad \text{for } x = 0. \qquad \text{(A.21)}$$

Thus,

$$E\{x'(T)M_i x(T)|x_0, r_0 = i\} \geq \exp(-\alpha_n(T - t_0))x_0'M_i x_0.$$

By the arguments of (A.20), we have

$$E\{x'(T)x(T)|x_0, r_0 = i\} \geq \exp(-\alpha_n(T - t_0))x_0'x_0.$$

If

$$T \leq t_0 + \frac{1}{\gamma}\ln\left(\frac{x_0'x_0}{\epsilon}\right)$$

then

$$E\{x'(T)x(T)|x_0, r_0 = i\} \geq \epsilon.$$

Thus, there exists no feedback control law which can drive $E\{x'(T)x(T)|x_0, r_0\}$ from all finite initial x_0 (note that although for some x_0 value which may lie in some subspace of $\mathbb{R}^n$ that is controllable, stochastic controllability requires that (6) be valid for any $x_0 < \infty$) to the ϵ-neighborhood of zero within time T. This contradicts the assumption of stochastic controllability of system (1)–(3). □

Proof of Corollary 2: Assume that system (1)–(3) is stochastically controllable. For each form $i \in \mathbb{S}$, there exists an L_i such that N_i and M_i in (7) satisfy (15). From (7), we have

$$\left(A_i - B_iL_i - \frac{1}{2}\lambda_i I\right)' M_i + M_i\left(A_i - B_iL_i - \frac{1}{2}\lambda_i I\right) = \sum_{j=1, j\neq i}^{s} \lambda_{ij}M_j - N_i = -\tilde{N}_i. \qquad \text{(A.22)}$$

We show that the pair (A_i, B_i) is deterministically controllable by contradiction. Assume that it is not. Then there is an eigenvalue μ, of $\left(A_i - B_iL_i - \frac{1}{2}\lambda_i I\right)$, which has a real part that cannot be made smaller than a given negative number $-\gamma/2$ by choice of L_i. Let z be the eigenvector associated with μ. From (A.22), we have

$$\mu + \bar{\mu} = 2\,\mathrm{Re}(\mu) = -\frac{\bar{z}'\tilde{N}_i z}{\bar{z}'M_i z}$$

where $\bar{z}'$ is the transpose of the conjugate of vector z and $\bar{\mu}$ is the conjugate of μ. Thus, we have, for any choice of L_i,

$$\frac{\mu_{\max}(\tilde{N}_i)}{\mu_{\min}(M_i)} = \max_z \left\{\frac{\bar{z}'\tilde{N}_i z}{\bar{z}'M_i z}\right\} = \max_z\{-2\,\mathrm{Re}(\mu)\} \leq \gamma.$$

This contradicts (15), and therefore the stochastic controllability assumption. □

Proof of Corollary 3: Here we only need to show sufficiency (necessity is given by Corollary 2). In (A.22), if i is an absorbing form, then (A.22) for this form i becomes

$$(A_i - B_iL_i)'M_i + M_i(A_i - B_iL_i) = -N_i$$

as for a single form system. Then, by Lemma 1, the necessary and sufficient condition (for stochastic controllability of this system) involving form i is that the pair (A_i, B_i) is controllable.

For a noncommunicating transient form i, $\lambda_{ij} \neq 0$ implies $\lambda_{ji} = 0$. That is, i does not communicate with other forms. Thus, if $\lambda_{ij}M_j \neq 0$ in the form i (A.22), then in the form j (A.22) we have $\lambda_{ji}M_i = 0$. That is, M_i is not coupled to the solution of M_j. Consequently, in solving for M_i, the right-hand side $\tilde{N}_i$ of (A.22) can be chosen arbitrarily. Thus, controllability of the pair (A_i, B_i) implies the controllability of $(A_i - \frac{1}{2}\lambda_i I, B_i)$; the latter can be used to show that (14) is satisfied for this form i (by the same proof steps as for the sufficiency of Lemma 1).□

Our definition of stochastic stabilizability only considers the case of constant (for each form value) feedback control laws. This is an accepted practice in defining stabilizability [29]. However, in defining controllability, the set of admissible controls in general is not restricted to constant (for each form value) feedback control laws. This is a serious restriction. If we use, instead, the admissible control class Ψ defined in Section I, the sufficient part of Theorem 2 is still valid. However, under this less-stringent requirement, we may not be able to obtain several useful results (e.g., for scalar case and for systems with only noncommunicating forms, the system is stochastically controllable if and only if each pair (A_i, B_i) is deterministically controllable).

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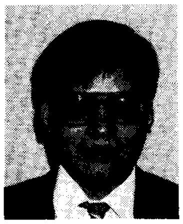

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