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Efficient payments: How much do they cost for the Central Bank?☆

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ABSTRACT

Previous works related to optimal denominations for coins and banknotes consider that the "principle of least effort" that defines an efficient payment is the most important criterion for two main reasons. Firstly, it is more convenient for transactors and, secondly, it limits the production costs of denominations incurred by the central bank. Exploiting production cost data for the U.S. currency system in 2010, we show using simulations that efficient payments actually increase the annual production costs of the Federal Reserve by $156 million. As a consequence, we raise a larger issue for central banks which consists in issuing an efficient denominational mix that is more convenient for transactors and that reduces the production costs of denominations.

1. Introduction

In recent years, abundant research has been devoted to the study of denominational structures of currency systems (Bouhadoui et al., 2011; Caianiello et al., 1982; Franses and Kippers, 2007; Lee et al., 2005; Sumner, 1993; Telser, 1995; Tschoegl, 1997; Van Hove, 2001; Van Hove and Heyndels, 1996; Wynne, 1997). Among the multiple properties of a currency system, the principle of least effort (PLE) is considered the most important.[2] This principle that defines an efficient payment states that the settlement of cash transactions should involve as few coins and notes as possible.

The preeminence of this principle, supported by many economists such as Boeschoten and Fase (1989), Eriksson and Kokkola (1993), Abrams (1995), Pedersen and Wagener (1996), Van Hove and Heyndels (1996) and Van Hove (2001), is justified by two main arguments. Firstly, the PLE states that it is more convenient for transactors given that it reduces the bulk and weight carried around by the cash-using public in turn limiting handling costs. Secondly, it keeps down the number of coins and notes in circulation and thus, so the reasoning goes, the production costs incurred by the central bank. Following this argument, it is therefore preferable for the central bank to opt for a currency system that limits the number of coins and notes used in transactions.

In this article, we demonstrate that the second argument is biased and that efficient payments increase the production costs incurred by the central bank. Our results tend therefore to support the idea that the private benefits of transactors emphasized in the economic literature can be undermined by the private costs of central banks. To prove this, we proceed in three stages. Firstly, we propose a general framework that links the costs of cash transactions to the production costs of the central bank. Secondly, we compare the costs of cash transactions using the PLE and a hypothetical cost-minimizing payment behavior named the "principle of least cost" (PLC). This latter minimizes the costs of cash transactions without considering the number of tokens exchanged in transactions; this model is only used to identify inefficient payments from the viewpoint of the PLE. Thirdly, we perform simulations on a set of cash transactions using production cost data for the U.S. currency system for the year 2010. The simulation results show that while the number of notes and coins used in transactions is certainly efficient (minimum) with the PLE, the costs of cash transactions are on average 24.2% greater than those obtained with the principle of least cost. Hence, while the PLE keeps down the total number of coins and notes in circulation it can also contribute to an increase in the costs of cash transactions and thus in the production costs of denominations incurred by the central bank. We precisely estimate the increase in the annual production cost to $156 million.

The remainder of the paper is structured as follows. In Section 2, we present the general framework and the cash payment behavior

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[2] Caianiello et al. (1982) point out two other elements, namely the surveyability (variety of denominations) and the compatibility with the decimal system.

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models. In Section 3, we describe the data used to perform simulations and comment on the results obtained. Finally, in Section 4, we discuss the implications of our results.

2. Model

In this section, we first present a general framework that links the costs of cash transactions to the production costs of denominations incurred by the central bank. Next, we present two models of cash payment behavior, namely the principle of least effort and the principle of least cost. Finally, we describe our comparison approach.

2.1. General framework

In a given economy, let set $\mathcal{D}$ be a distribution composed of N_T cash transactions. The distribution $\mathcal{D}$ represents the cash transactions made by the public during a year. To pay in cash the N_T transactions, the agents use a currency system composed of J tokens of face values $v(j)$ with $j \in 1, ..., J$. Regardless of, for the moment, the way people use the denominations in transactions (see below), we denote by k one of the $K(x)$ solutions to pay an amount x. Following this combination, the amount x is paid by exchanging $n_k(x, j)$ token(s) for each denomination j such that:

$$x = \sum_j n_k(x, j) \cdot v(j). \tag{1}$$

The integer $n_k(x, j)$ is set positive when the money is given by the consumer to the merchant and negative when it is a return of change.

Considering all the possible combinations, the average number of times a denomination j is involved in a transaction x is denoted by $a(x, j)$ and called the frequency of use of the denomination j for the amount x:

$$a(x, j) = \frac{1}{K(x)} \cdot \sum_k |n_k(x, j)|. \tag{2}$$

Using Eq. (2), we can define the average frequency of use of a denomination j over the distribution $\mathcal{D}$ as:

$$a(j) = \frac{1}{N_T} \cdot \sum_{x \in \mathcal{D}} a(x, j). \tag{3}$$

Let us now define the usage costs of denominations. To begin with, we assume that the production cost $c_p(j)$ of a denomination includes all the costs and expenses for producing, marketing and distributing coins and notes, and we introduce the depreciation rate per use, $\delta(j)$, of a denomination that captures its deterioration after each use. Multiplying $c_p(j)$ and $\delta(j)$, we can then write the usage cost, $c_u(j)$, of a denomination as:[3]

$$c_u(j) = c_p(j) \cdot \delta(j). \tag{4}$$

The depreciation rate per use of a denomination, $\delta(j)$, depends primarily on the resistance of the manufacturing technology. It can be expressed as a function of the life span, $d(j)$, and the annual velocity of circulation of a denomination, $q_a(j)$, that refers to the average number of times per year a circulating token j is involved in a cash transaction:[4]

$$\delta(j) = \frac{1}{d(j) \cdot q_a(j)}. \tag{5}$$

Likewise, the annual velocity of circulation can be defined as the ratio of the number of uses per year of all the circulating tokens, j, measured with the term $(N_T \cdot a(j))$, and their circulating volume $N_c(j)$:

$$q_a(j) = \frac{N_T \cdot a(j)}{N_c(j)}. \tag{6}$$

Therefore, replacing Eq. (6) in Eq. (5), we obtain:

$$\delta(j) = \frac{N_c(j)}{d(j) \cdot N_T \cdot a(j)}. \tag{7}$$

The central bank is generally responsible of the processing of the currency in circulation. During this operation, the substandard tokens are withdrawn and replaced by new ones. The volume, $N_r(j)$, of tokens of denomination j replaced each year is determined by $N_c(j)$ and $d(j)$:

$$N_r(j) = \frac{N_c(j)}{d(j)}. \tag{8}$$

Then, replacing Eq. (8) with Eq. (7), we have:

$$\delta(j) = \frac{N_r(j)}{N_T \cdot a(j)}. \tag{9}$$

We finally obtain the expression of the usage cost of a denomination, $c_u(j)$, after replacing Eq. (9) with Eq. (4):

$$c_u(j) = \frac{c_p(j) \cdot N_r(j)}{N_T \cdot a(j)}. \tag{10}$$

Using $c_u(j)$, we can finally define the cost of a cash transaction as follows:

$$c_u(x) = \sum_j a(x, j) \cdot c_u(j). \tag{11}$$

The last step of the general framework consists in linking the cost of cash transactions to the production costs of denominations incurred by the central bank. The annual production cost of currency, C_r, incurred by the central bank is by definition related to the new tokens replaced each year, *i.e.*:

$$C_r = \sum_j N_r(j) \cdot c_p(j). \tag{12}$$

Rearranging Eq. (10) and replacing in Eq. (12), one can write:

$$C_r = N_T \cdot \sum_j a(j) \cdot c_u(j). \tag{13}$$

Finally, using Eq. (3) then Eq. (11) with Eq. (13), we obtain:

$$C_r = \sum_x c_u(x). \tag{14}$$

Eq. (14) shows that the costs of cash transactions $c_u(x)$ are directly related to the production costs of denominations incurred by the central bank C_r. As a result, the latter are affected by the way people use denominations in transactions. In the next part, we introduce two models of cash payment behavior.

2.2. Models of cash payment behavior

This part aims at formalizing and comparing two models of cash payment behavior. The first is the "principle of least effort" that we extend to account for the usage costs of denominations. The second one is a hypothetical cost-minimizing model called the "principle of

[3] For instance, if the production cost of a banknote is $c_p(j) = 0.1\$$ and the depreciation rate per use is $\delta(j) = 5\%$ then the usage cost is $c_u(j) = 0.005\$$.

[4] For instance, if $d(j) = 2$ years and $q_a(j) = 5$ uses per year, the depreciation rate per use is $\delta(j) = \frac{1}{2 \times 5} = 10\%$ per use.

least cost" that we use as a benchmark to assess the cost efficiency of the principle of least effort.

2.2.1. The principle of least effort

The principle of least effort (PLE) was introduced by Caianiello et al. (1982) and subsequently refined by Cramer (1983). Following this principle, a consumer and a merchant use the minimum number of coins and notes to pay a given amount of cash. More formally, an amount x is paid efficiently by exchanging $n^{PLE}(x, j)$ tokens of each denomination j with $j \in 1, ..., J$:

$$x = \sum_j n^{PLE}(x,j) \cdot v(j), \tag{15}$$

such that the number of coins and notes exchanged $n^{PLE}(x)$ is minimum, with:

$$n^{PLE}(x) = \sum_j \left| n^{PLE}(x,j) \right|. \tag{16}$$

Absolute values in Eq. (16) indicate that overpayment and return of change are allowed.[5]

As noted in Eq. (1), several solutions can exist to pay a given amount x and give rise, therefore, to several costly combinations. The cost of a combination k can be written as follows:

$$c_u^{PLE}(x)_k = \sum_j \left| n_k^{PLE}(x,j) \right| \cdot c_u^{PLE}(j). \tag{17}$$

Given that an amount can be paid with different costly combinations, we denote respectively by $c_u^{PLE}(x)_{min}$ and $c_u^{PLE}(x)_{max}$ the minimum and the maximum costs for the transaction x. In addition, using the average frequencies of use of the denominations, we can measure the average cost of a cash transaction as follows (cf. Eq. (11)):

$$c_u^{PLE}(x) = \sum_j a^{PLE}(x,j) \cdot c_u^{PLE}(j). \tag{18}$$

2.2.2. The principle of least cost

By analogy to the principle of least effort, we propose an alternative model called the "principle of least cost" (PLC). This model combines a number of coins and notes to calculate the minimum cost of a cash transaction and, consequently, disregards the number of tokens used in the transaction. This model is only used as a reference to calculate inefficient payments from the viewpoint of the PLE. Indeed, since the PLC is not sensitive to the number of tokens used in transactions, it provides inefficient solutions compared to the PLE. We can therefore use the PLC as a reference to compare the costs of cash transactions obtained with the PLE. It is worth noting that the PLC could constitute what could be optimal from the central bank viewpoint since the costs of cash transactions are minimal and thus the production costs.

Following the PLC, an amount x is paid by exchanging $n^{PLC}(x, j)$ token(s) of each denomination j:

$$x = \sum_j n^{PLC}(x,j) \cdot v(j), \tag{19}$$

such that the cost of the cash transaction, $c_u^{PLC}(x)$, is minimum:

$$c_u^{PLC}(x) = \sum_j \left| n^{PLC}(x,j) \right| \cdot c_u^{PLC}(j). \tag{20}$$

It is important to outline that several optimal combinations are also possible with the PLC. However, on the contrary to the PLE, they all have, by definition, the same cost.

3. Simulations: the U.S. currency system

This part aims at comparing the costs of cash transactions when the public is supposed to follow the PLE or PLC. To do that, we exploit the production volumes and cost data provided by the Federal Reserve to measure the usage costs of U.S. denominations and then simulate a series of payments with the U.S. currency system.

3.1. The usage costs of U.S. denominations

According to Eq. (10), measuring the usage cost of denominations requires to obtain the unit production costs, the annual replacement volumes, the average frequencies of use and an estimation of the number of cash transactions performed during a year.

To begin with, the production costs of denominations are detailed in the annual reports of the United States Mint and the annual "New Currency Budgets" of the Federal Reserve Board.[6] We observe that some denominations are produced in different series such as the \$100 banknote with different unit production costs. For instance, in 2010, the \$100 banknote was produced either in Series-1996 at a unit production cost of \$0.097 and for a total number of units of about 108.8 millions or in Series-2004 at \$0.134 for a total number of 2,751.2. To account for this problem, we calculate a weighted average production cost which, in the case of the \$100 denomination, amounts to \$0.110. Table 1 summarizes the production costs per denomination for the year 2010.

Next, the Federal Reserve kindly provided us with detailed information on the volume of banknotes replaced in year 2010. For the coins, we rely on the annual production volume provided by the Mint for the year 2010 assuming that all coins produced this year are put into circulation.[7] The volumes obtained are also summarized in Table 1.[8]

Turning to the annual number of cash transactions, the only study that provides such an estimation is, to the best of our knowledge, that of Gerdes et al. (2005). According to the authors, the total number of cash transactions in the U.S. amounts approximately to 100 billion. We then use this reference as a rough approximation.

Finally, the remaining data required to calculate the usage costs of denominations are the average frequencies of use $a(j)$ (cf. Eq. (10)) which are closely related to the cash payment behavior and to the distribution of the cash transactions in the U.S. economy. Regarding the cash payment behavior, Franses and Kippers (2007) have clearly shown that the PLE

[5] The original payment algorithm formalized by Cramer (1983) assumes that economic agents have available all possible notes and coins. If this assumption seems to be realistic in the case of merchants, it does not necessary hold for consumers. Franses and Kippers (2007) have proposed a statistical model to account for the unavailability of some denominations in wallets. Using survey data on payments realized by Dutch consumers, the authors show that "all individuals are inclined to make efficient payments, irrespective of whether or not the wallet facilitates the unconstrained efficient payments".

[6] Let us note that the U.S. production costs of denominations are probably overstated with respect to other countries mainly because of the overuse of certain denominations (1\$), the existence of different series of the same denominations (1\$) and numismatic considerations. However, as we will see later, our objective is not to comment on the formation of production costs but just to take them as raw data for comparison. As we will explain later, we mainly need to show that there are significant cost differences between denominations to highlight that the principle of least effort increases the production costs incurred by the central bank.

[7] The existence of the penny coin has been criticized over the last years and two bills introduced in the U.S. Congress aimed at ceasing the production of pennies. One of the arguments is that the production cost of the penny exceeds his face value. However, the opponents of this reform argue that prices could rise and induce a revision of sales tax rules. This debate is beyond the scope of this article although the presence of the penny coin affects the formation of prices and cash payments. We will however consider in the rest of the paper that the effect of the existence of the penny coin on the formation of prices and cash payments is negligible.

[8] We do not include the \$0.50 denomination and the \$1 coin because they do not circulate widely.

Table 1
The unit production costs and the annual replacement volumes of U.S. denominations in 2010.

Denomination	Unit production cost (in $)	Annual replacement volume (in millions)
$v(j)$	$c_p(j)$	$N_r(j)$
$0.01	0.0179	4010.83
$0.05	0.0922	490.56
$0.10	0.0569	1119.00
$0.25	0.1278	347.00
$1	0.0481	2622.04
$5	0.0776	673.01
$10	0.0789	484.26
$20	0.0851	1270.22
$50	0.0851	104.92
$100	0.1100	785.54

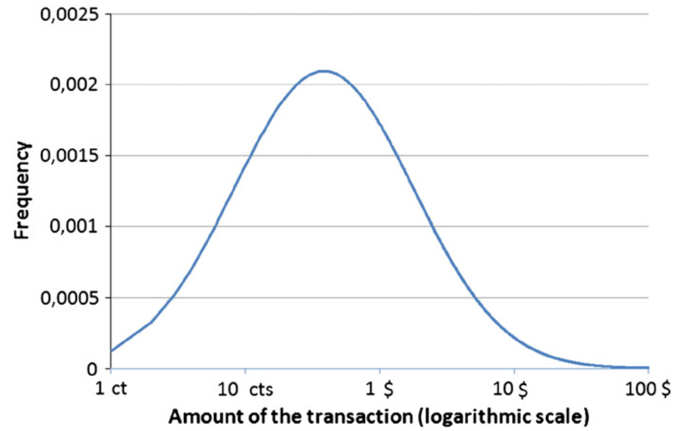

Fig. 1. Lognormal distribution of cash transactions as a function of the amounts.

constitutes a reasonable approximation of the public payment behavior. Likewise, it is well established in the economic literature since Boeschoten and Fase (1989), that a lognormal distribution better fits the observed cash transactions. As a consequence, we set up a lognormal distribution of cash transactions characterized by an average amount of $11.52 (Garcia-Swartz et al., 2006) and a standard deviation of $30.[9] Fig. 1 depicts the lognormal distribution arbitrarily limited to payments below $100.[10]

We now have all the data to calculate the usage costs of the U.S. denominations using the PLE. The results are provided in Table 2.

3.2. Results

Several comments can be made about the simulations.

Firstly, the simulations show that 48.8% of the transactions of the distribution admit more than one efficient combination with the PLE. Indeed, as noted in the first section, several amounts can be paid in multiple efficient ways. For instance, the amount $26.20 is payable with two efficient combinations both exchanging five tokens, namely $20 + $5 + $1 + $0.25 − $0.05 and $20 + $5 + $1 + (2 × $0.10). Now, the efficient combinations are not equivalent from a cost perspective because the usage costs of denominations are not homogeneous (column 3 of Table 2) and then some combinations clearly increase the costs of cash transactions by using costly denominations. For instance, the costs of the two combinations mentioned in the above example are respectively $0.0107 and $0.0113. Globally, over the distribution, the average gap between the most and the least costly PLE combinations, $(c_u^{PLE}(x)_{max} - c_u^{PLE}(x)_{min})$, amounts to $0.0016 per transaction.

Secondly, we simulated the same pattern of cash payments by assuming this time that the people use the PLC. Fig. 2 compares the frequencies of use of the denominations obtained in the case of the PLE and PLC. We note that some of the average frequencies equal zero ($20 and $100) in the case of the PLC. This finding is mainly due to the trade-off operated by the PLC. For example, the PLC prefers to use 2 × $10 instead of a $20 banknote although it involves an additional token as the usage cost of a $20 note is greater than the usage costs of two $10 notes $(c_u(\$20) > 2 \times c_u(\$10))$. This result also indicates that the truncature of the cash payment distribution is not a serious problem for our concern. Indeed, as indicated above, if we had included amounts above $100, they would necessarily have included one or several $100 notes. Now, as the usage cost of a $100 note is higher than the usage costs of two $50 notes, the gap in the average cost per transaction would even have been greater between the PLE and PLC.

Overall, compared to the PLC, the PLE reduces the use of tokens in transactions by 6.1%, but costs in turn around a quarter more (cf. Table 3). Going into details on the cost gap between the PLE and PLC, it turns out that 61.7% of the PLE payments are not cost efficient. Indeed, as indicated previously, when comparing the frequencies of use, the PLC efficient combinations include more tokens than the PLE ones. As a result, they are not considered by the PLE. To follow the previous example related to the amount $26.20, the PLC prefers to use the following combination (2 × $10) + $5 + $1 + $0.25 − $0.05 that involves six tokens at a cost of about $0.0081, the latter being clearly lower than the two PLE efficient combinations ($0.0107 and $0.0113).

Finally, we can answer to the main question of this paper: How much do efficient payments cost for the Federal Reserve? To do that, we measure the impact of the model of cash payment behavior on the production costs of denominations using Eq. (13). We find that efficient cash payments (PLE) induce an increase in the annual production cost for the Federal Reserve by $156 million.

4. Conclusion

The issue of the cost of cash has become today an important topic for economists, banks and monetary authorities. The different central banks all over the world have indeed tried on numerous occasions to reduce the cost of banknote production either by withdrawing a denomination (Chen, 1976) or by introducing coins in place of banknotes (Lotz and Rocheteau, 2004) or, finally, by introducing a new technology such as

Table 2
Average frequencies of use according to the PLE and usage costs of U.S. denominations in 2010.

Denomination	Average frequency of use with the PLE	Usage cost (in $)
$v(j)$	$a^{PLE}(j)$	$c_u(j)$
$0.01	1.24	0.00058
$0.05	0.36	0.00124
$0.10	0.54	0.00117
$0.25	0.86	0.00051
$1	1.21	0.00110
$5	0.43	0.00139
$10	0.23	0.00194
$20	0.18	0.00648
$50	0.05	0.00143
$100	0.01	0.03704

[9] In appendix, we study how the variation in the results is sensitive to the average amount and the standard deviation of the value of a cash transaction.

[10] According to Garcia-Swartz et al. (2006), less than 10% of the U.S. transactions at point-of-sale are paid in cash. We then arbitrarily decided to truncate the lognormal distribution of cash transactions to transactions below $100. However, as we will explain later, including large-value cash transactions (above $100) would not change the results since the usage cost of a $100 note is greater than the usage costs of two $50 notes. As a result, the gap in the average cost per transaction would be higher between the PLE and PLC.

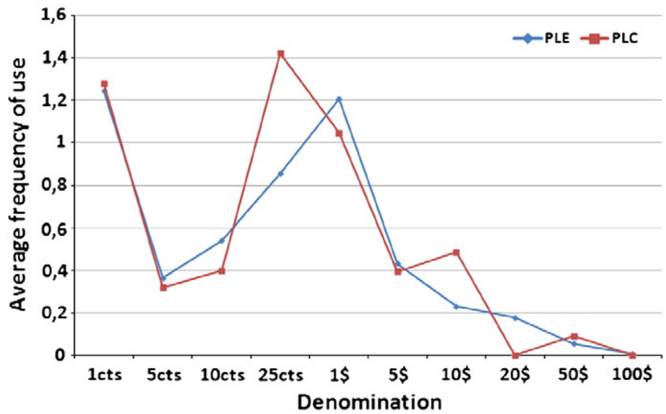

Fig. 2. Average frequencies of use of denominations with the PLE and PLC.

Table 3
Efficient payments and cost of transactions with the PLE and PLC.

	PLE	PLC	Gap
Average number of tokens exchanged per transaction	4.97	5.29	6.1%
Average cost per transaction	$0.0065	$0.0049	− 24.2%

the polymer one (Bouhdaoui et al., 2012). The commercial banks have also encouraged the public to use electronic payment instruments supposedly less costly for the society. These strategies seem to be supported by empirical studies that conclude that debit card payments are socially less costly than paper-based payments (Guibourg and Seggendorff, 2007).

Our paper contributes to this topic in two different ways. First, our results show in the case of the U.S. currency system that efficient payments can increase the costs of cash transactions and then the annual production costs of denominations incurred by the Federal Reserve. The main argument is that the usage costs of denominations significantly differ and that it can be efficient from a cost perspective to use more cheaper coins and notes in transactions. Now, by definition, the principle of least effort can fail to reach this objective.

Second, our paper provides not only a simple method to assess the cost efficiency of a denomination in a currency system but also some ways to restore the overall efficiency.

On the one hand, at the level of each denomination, our model provides an original approach to measure the usage cost of a denomination (Eq. (10)). This measure is more useful for a central bank than the mere knowledge of the unit production cost as a denomination can certainly have a high unit production cost but be widely used in transactions; as a result, its usage cost is low. This is the case for example for the 25 cents denomination (comparison of Tables 1 and 2). Now, by measuring and comparing usage costs of denominations, a central bank can determine the denominations that quickly deteriorate with regard to their production cost and come up with solutions to reduce their usage cost. These concerns are lively in a large number of countries. For instance, in Europe, some countries have decided to stop issuing the lowest denominations (1 and 2 cents) as they are supposed to have a limited utility in transactions. Our model can then accurately measure the effects of such decision on the use of related denominations and on the induced changes in production costs for the central bank.

On the other hand, at the level of the overall cost efficiency of a currency system, assuming that the public behaves according to the principle of least effort, which is not a restrictive assumption (Franses and Kippers, 2007), central banks can be interested in designing a currency system so as to reduce the production costs of denominations according to this payment behavior. In this perspective, our paper introduces a hypothetical cash payment behavior, the principle of least cost that can be used as a benchmark by central banks to measure the “extra cost” of their currency system. As a consequence, our paper raises a larger issue for monetary authority: making a denominational mix more convenient for transactors while reducing the resulting production costs. This issue of the cost efficiency of a denominational structure has gone largely unresearched (Massoud, 2005) even though recent monetary contributions attempt to tackle this issue (Bouhdaoui et al., 2011; Lee et al., 2005).

Table 4
Gap in the average number of tokens exchanged per transaction between PLE and PLC (in %): scenarios.

Average cash transaction (standard deviation)	$20	$30	$40
$10	5.9	5.7	6.2
$11.52	6.0	5.9	5.8
$13	6.2	6.1	6.1

Table 5
Gap in the average cost per transaction between PLE and PLC (in %): scenarios.

Average cash transaction (standard deviation)	$20	$30	$40
$10	19.2	20.6	20.7
$11.52	20.9	22.4	22.7
$13	22.5	24.1	24.3

Appendix A

In this part, we test the sensitivity of the results to a variation in the parameters of the model. In particular, we simulated in our benchmark scenario a lognormal distribution of cash transactions characterized by an average cash transaction of $11.52 and a standard deviation of $30. We test hereafter different scenarios. To begin with, we assume a change of the average transaction upward and downward of about 10%, *i.e.* $10 and $13. After, we simulate a change of the standard deviation, from $20 to $40. Finally, we combine all the assumptions. Tables 4 and 5 below summarize the results. We globally note that the gaps in the average number of tokens exchanged per transaction and in the average cost per transaction are very small and pretty close to our benchmark scenario. In particular, the highest gap in the average cost per transaction (in %) with respect to our benchmark scenario amounts to 3.2% (scenario characterized by an average cash transaction of $10 and a standard deviation of $20). We can conclude therefore that our results are reasonably stable and not too much sensitive to the main parameters of our model.

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