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Brief paper

Robust mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control of networked control systems with random time delays in both forward and backward communication links[☆]

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ABSTRACT

This paper is concerned with the two-mode-dependent robust control synthesis of networked control systems where random delays existing in both forward controller-to-actuator (C–A) and feedback sensor-to-controller (S–C) communication links are modeled as Markov chains. The output feedback controller is designed to depend on the current S–C delay and the previous C–A delay. Then, the closed-loop system is formulated as a special jump linear system. The generalized definitions of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms for such underlying special systems are proposed. Further, the two-mode-dependent robust $\mathcal{H}_2$ and robust mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control design methods for NCSs are developed. The design examples illustrate the effectiveness of the proposed methods.

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1. Introduction

Networked control systems (NCSs) have advantages such as low cost, easy diagnosis, and high mobility over the traditional control system. Hence, NCSs have been finding many industrial applications in automobiles, manufacturing plants, and aircrafts; see, e.g., Nilsson (1998), Seiler and Sengupta (2005), Shi and Fang (2010) and Tsai and Ray (1999), and the references therein. The introduction of communication networks into control systems also brings some design constraints such as network-induced delays, packet dropouts, and quantization errors. It is well known that the time delays and packet dropouts can degrade the system performance or even cause instability of control systems. Therefore, how to handle the network-induced delays and packet dropouts has attracted much attention. To model the time delays and/or packet loss process, there are two basic approaches. The first approach is the stochastic method which assumes that the time delay and/or packet loss processes follow certain probability distributions, e.g., Markov chain (Nilsson, 1998) and Bernoulli process (Seiler & Sengupta, 2005; Shi, Fang, & Yan, 2009); the second approach is the deterministic method, which places bounds on time delays and/or packet losses or specifies them in a time average sense (Lin & Antsaklis, 2005).

In a general NCS as shown in Fig. 1, network-induced delays exist on both the *forward* controller-to-actuator (C–A) and the *feedback* sensor-to-controller (S–C) communication links. Various methodologies have been proposed in the literature for the controller design considering the network-induced delays and/or packet dropouts. The controller can be time-invariant (mode independent) or depend on S–C and/or C–A delays (one mode dependent or two mode dependent). The mode-independent controller does not depend on either S–C or C–A delays. Since the controller is mode independent, the S–C and C–A delays can be lumped together as a whole, and therefore, the methods for time-delayed systems can be applied to NCSs. Examples of such controller design are the research work in Gao and Chen (2008), Xiao, Hassibi, and How (2000) and Xiong and Lam (2007). The one-mode-dependent controller usually depends on S–C delays. Then the closed-loop system was formulated as a jump linear system and the stabilization and control synthesis can be solved under the framework of jump linear systems as in Seiler and Sengupta (2005) and Xiao et al. (2000). The two-mode-dependent controller refers to that the controller depends on both S–C and C–A delays. Zhang, Shi, Chen, and Huang (2005) developed a new state feedback controller that simultaneously depends on both S–C and C–A delays modeled by Markov chains, and provided the sufficient and necessary condition to guarantee stochastic stability. In Huang and Nguang (2008), the Markov processes were used to model the random network-induced delays and a two-mode-dependent state feedback controller was designed for continuous-time NCSs based on the Lyapunov–Razumikhin method. Recently, in Shi and Yu (2009), the S–C and C–A delays, modeled by two Markov chains, were simultaneously incorporated into the controller design in a

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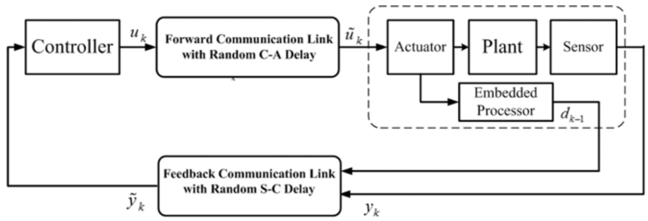

Fig. 1. Diagram of a networked control system.

general and practical way, and the output feedback control design was developed.

However, to the best of our knowledge, the two-mode-dependent robust control synthesis problem for NCSs has not been addressed, which is the focus of this paper. A closer scrutiny of incorporating both forward C–A and feedback S–C delays into controller design necessarily requires the consideration on the availability of the C–A delay information at the controller node in practical NCSs. In comparison with the aforementioned references, it is a worthwhile effort to examine how to make full use of the available forward C–A and feedback S–C delay information in designing the two-mode-dependent controllers for NCSs. First, to be of any practical use in applications, the current forward C–A delay d_k may not be available at the controller node at time instant k, because the transmission of such delay information is also subject to the network-induced delay. Second, owing to the use of Markov chain models for the delays, and the strategy of incorporating both S–C and the most recent available C–A delays into the design, the well-established results in MJLSs (Costa & Marques, 2000; Seiler & Sengupta, 2003) cannot be directly applied. As will be seen in later sections, the inclusion of the most recent available C–A delay renders difficulty in analyzing the underlying special jump system. Third, the robust $\mathcal{H}_2$ and robust mixed $\mathcal{H}_2/\mathcal{H}_\infty$ two-mode-dependent control for NCSs have not been reported in the literature. Thus the development of such two-mode-dependent robust $\mathcal{H}_2$ and robust mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control for NCSs is both practically necessary and theoretically interesting.

The remainder of this paper is organized as follows. Section 2 describes the systems we deal with and states the objectives of this work. Section 3 proposes the robust $\mathcal{H}_2$ and robust mixed $\mathcal{H}_2/\mathcal{H}_\infty$ two-mode-dependent control design for NCSs, respectively. We give an illustrative design example in Section 4. Finally, the concluding remarks and future work are addressed in Section 5.

Notation: The superscripts "T" and "−1" stand for matrix transposition and matrix inverse, respectively. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and the notation $P > 0$ means that P is real symmetric and positive definite. diag$\{\cdots\}$ stands for a block-diagonal matrix and tr$\{\cdot\}$ means the trace of a matrix. $\|\cdot\|_2$ refers to the Euclidean norm for vectors and induced 2-norm for matrices. $\mathcal{E}(\cdot)$ stands for the mathematical expectation operator. $l_2[0,\infty)$ is the space of a square summable vector sequence over $[0,\infty)$.

2. Problem formulation

Consider the NCS setup in Fig. 1. The uncertain discrete-time plant model is

$$x(k+1) = (A+\Delta A)x(k) + (B+\Delta B)\tilde{u}(k) + J\omega(k),$$
$$y(k) = Cx(k), \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $\tilde{u}(k) \in \mathbb{R}^m$ is the control input, $y(k) \in \mathbb{R}^p$ is the controlled output, $\omega(k) \in \mathbb{R}^l$ is the exogenous disturbance signal which belongs to $l_2[0,\infty)$, A, B, C, and J are known real constant matrices with appropriate dimensions. $\Delta A(k)$ and $\Delta B(k)$ are unknown matrices representing the time-varying norm-bounded uncertainties that satisfy the following condition:

$$\begin{bmatrix}\Delta A(k) & \Delta B(k)\end{bmatrix} = G_1 \Delta_u(k) \begin{bmatrix}U_1 & U_2\end{bmatrix}. \tag{2}$$

Here, G_1, U_1, and U_2 are known real constant matrices with appropriate dimensions, and $\Delta_u(k)$ is the unknown time-varying matrix function subject to $\Delta_u(k)^{\mathrm{T}}\Delta_u(k) \le I$. Random delays in S–C and C–A links as shown in Fig. 1 are assumed to be bounded and multiples of the sampling period. Here, $0 \le \tau_k \le \tau$ represents the S–C delay and $0 \le d_k \le d$ stands for the C–A delay. Further, τ_k and d_k are modeled as two homogeneous Markov chains that take values in $\mathcal{M} = \{0,1,\ldots,\tau\}$ and $\mathcal{N} = \{0,1,\ldots,d\}$, and their transition probability matrices are $\Lambda = [\Lambda_{ij}]$ and $\Pi = [\Pi_{rs}]$, meaning that τ_k and d_k jump from mode i to j and from mode r to s, respectively, with probabilities Λ_{ij} and Π_{rs}, which are defined by

$$\Lambda_{ij} = \Pr(\tau_{k+1} = j|\tau_k = i), \qquad \Pi_{rs} = \Pr(d_{k+1} = s|d_k = r)$$

with the constraints $\Lambda_{ij}, \Pi_{rs} \ge 0$ and

$$\sum_{j=0}^{\tau} \Lambda_{ij} = 1, \qquad \sum_{s=0}^{d} \Pi_{rs} = 1$$

for all $i, j \in \mathcal{M}$ and $r, s \in \mathcal{N}$.

Remark 1. The network-induced delays and packet dropouts may lead to the disorder (out of order) of arriving packets. The Markov chain model assumes that the most recent data is used at the controller node. A buffer could be used to store the latest data that has the most recent time stamp. When a new packet arrives at the buffer, if the time stamp of the data is greater than that of the data stored in the buffer, the new data will be stored in the buffer and applied.

As shown in Shi and Yu (2009), the controller can be designed based on the S–C delay τ_k and the previous C–A delay $d_{k-\tau_k-1}$. Hence, the two-mode-dependent output feedback controller has the following form:

$$z(k+1) = F(\tau_k, d_{k-\tau_k-1})z(k) + G(\tau_k, d_{k-\tau_k-1})\tilde{y}(k),$$
$$u(k) = H(\tau_k, d_{k-\tau_k-1})z(k) + T(\tau_k, d_{k-\tau_k-1})\tilde{y}(k), \tag{3}$$

where $z(k) \in \mathbb{R}^n$ is the state vector of the output feedback controller; F, G, H, and T are appropriately dimensioned matrices to be designed. In total, there are $4(\tau+1)(d+1)$ matrices to be designed.

At sampling time k, if we augment the state and output variables of the plant as

$$\tilde{X}(k) = \begin{bmatrix}x(k)^{\mathrm{T}} & y(k-1)^{\mathrm{T}} & y(k-2)^{\mathrm{T}} & \cdots & y(k-\tau)^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}},$$

and consider that $\tilde{y}(k) = y(k-\tau_k)$, then we have

$$\tilde{X}(k+1) = \tilde{A}\tilde{X}(k) + \tilde{B}\tilde{u}(k) + \tilde{J}_1\omega(k),$$
$$\tilde{y}(k) = \tilde{C}_1(\tau_k)\tilde{X}(k), \tag{4}$$

where

$$\tilde{A} = \begin{bmatrix} A+\Delta A & 0 & \cdots & 0 & 0 \\ C & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} B+\Delta B \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

$$\tilde{J}_1 = \begin{bmatrix}J^{\mathrm{T}} & 0 & \cdots & 0 & 0\end{bmatrix}^{\mathrm{T}},$$

$$\tilde{C}_1(\tau_k) = \begin{cases} \begin{bmatrix}C & 0 & \cdots & 0 & 0 & \cdots & 0\end{bmatrix}, & \text{for } \tau_k = 0, \\ \underbrace{\begin{bmatrix}0 & \cdots & 0 & I & 0 & \cdots & 0\end{bmatrix}}_{(1+\tau_k)\text{th block being identity}}, & \text{for } \tau_k > 0. \end{cases}$$

Similarly, at sampling time k, augment the state and output variables of the controller as

$$\tilde{Z}(k) = \begin{bmatrix} z(k)^{\mathrm{T}} & u(k-1)^{\mathrm{T}} & u(k-2)^{\mathrm{T}} & \cdots & u(k-d)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$

then

$$\tilde{Z}(k+1) = \tilde{F}(\tau_k, d_{k-\tau_k-1})\tilde{Z}(k) + \tilde{G}(\tau_k, d_{k-\tau_k-1})\tilde{y}(k),$$
$$\tilde{u}(k) = \tilde{H}(\tau_k, d_k, d_{k-\tau_k-1})\tilde{Z}(k) + \tilde{T}(\tau_k, d_k, d_{k-\tau_k-1})\tilde{y}(k), \quad (5)$$

where

$$\tilde{F}(\tau_k, d_{k-\tau_k-1}) = \begin{bmatrix} F(\tau_k, d_{k-\tau_k-1}) & 0 & \cdots & 0 & 0 \\ H(\tau_k, d_{k-\tau_k-1}) & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix},$$

$$\tilde{G}(\tau_k, d_{k-\tau_k-1}) = \begin{bmatrix} G(\tau_k, d_{k-\tau_k-1}) \\ T(\tau_k, d_{k-\tau_k-1}) \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

$$\tilde{H}(\tau_k, d_{k-\tau_k-1}, d_k) = \begin{cases} \begin{bmatrix} H(\tau_k, d_{k-\tau_k-1}) & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}, & \text{for } d_k = 0, \\ \underbrace{\begin{bmatrix} 0 & \cdots & 0 & I & 0 & \cdots & 0 \end{bmatrix}}_{(1+d_k)\text{th block being identity}}, & \text{for } d_k > 0, \end{cases}$$

$$\tilde{T}(\tau_k, d_{k-\tau_k-1}, d_k) = \begin{cases} T(\tau_k, d_{k-\tau_k-1}), & \text{for } d_k = 0, \\ 0, & \text{for } d_k > 0. \end{cases}$$

Combining (4) and (5), and defining the state variable as

$$X(k) = \begin{bmatrix} \tilde{X}(k)^{\mathrm{T}} & \tilde{Z}(k)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$

we have the closed-loop system dynamics as follows:

$$X(k+1) = \left[\bar{A} + \bar{B}K(\tau_k, d_{k-\tau_k-1}, d_k)\bar{C}(\tau_k)\right]X(k) + \tilde{J}\omega(k),$$
$$y(k) = \tilde{C}X(k), \quad (6)$$

where

$$\bar{A} = \begin{bmatrix} \tilde{A} & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 & \tilde{B} \\ I & 0 \end{bmatrix}, \quad \bar{C}(\tau_k) = \begin{bmatrix} 0 & I \\ \tilde{C}_1(\tau_k) & 0 \end{bmatrix},$$

$$K(\tau_k, d_{k-\tau_k-1}, d_k) = \begin{bmatrix} \tilde{F}(\tau_k, d_{k-\tau_k-1}) & \tilde{G}(\tau_k, d_{k-\tau_k-1}) \\ \tilde{H}(\tau_k, d_{k-\tau_k-1}, d_k) & \tilde{J}(\tau_k, d_{k-\tau_k-1}, d_k) \end{bmatrix},$$

$$\tilde{J} = \begin{bmatrix} J^{\mathrm{T}} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}^{\mathrm{T}},$$
$$\tilde{C} = \begin{bmatrix} C & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Eq. (6) represents a discrete-time jump linear system. For the ease of the presentation, when the system is in mode $i \in \mathcal{M}$ and $r \in \mathcal{N}$ (i.e., $\tau_k = i$, $d_{k-\tau_k-1} = r$), $F(\tau_k, d_{k-\tau_k-1})$, $G(\tau_k, d_{k-\tau_k-1})$, $H(\tau_k, d_{k-\tau_k-1})$, and $T(\tau_k, d_{k-\tau_k-1})$ will be denoted as $F(i, r)$, $G(i, r)$, $H(i, r)$, and $T(i, r)$, respectively.

Note that the matrices $\bar{A}$ and $\bar{B}$ are of the following form

$$\bar{A} = A_{\mathrm{Aug}} + \Delta A_{\mathrm{Aug}} = A_{\mathrm{Aug}} + \tilde{G}_1 \Delta_u(k) U_1 I_2^{\mathrm{T}}, \quad (7a)$$
$$\bar{B} = B_{\mathrm{Aug}} + \Delta B_{\mathrm{Aug}} = B_{\mathrm{Aug}} + \tilde{G}_1 \Delta_u(k) U_2 I_3^{\mathrm{T}}, \quad (7b)$$

where

$$A_{\mathrm{Aug}} = \left[\begin{array}{ccccc|ccc} A & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ C & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & I & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & 0 & \cdots & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{array}\right]$$

$$B_{\mathrm{Aug}} = \left[\begin{array}{ccc|c} 0 & \cdots & 0 & B \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \vdots \\ \vdots & \ddots & \vdots & 0 \\ 0 & \cdots & 0 & 0 \\ \hline I & 0 & 0 & 0 \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & I & 0 \end{array}\right]$$

$$\tilde{G}_1 = \begin{bmatrix} G_1^{\mathrm{T}} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}^{\mathrm{T}},$$
$$I_2 = \begin{bmatrix} I & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}^{\mathrm{T}},$$
$$I_3 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & I \end{bmatrix}^{\mathrm{T}}.$$

Then, it is clear that the uncertainties in the augmented system in (6) are also norm bounded.

Remark 2. If we augment the possible network-induced delay values at the same time by defining a vector-valued variable as $\Phi_k = \begin{bmatrix} \tau_k & d_k & d_{k-1} & d_{k-2} & \cdots & d_{k-\tau-1} \end{bmatrix}$, the closed-loop system can be transformed to a standard MJLS. However, this approach would bring two difficulties for the controller design: (1) It is hard to determine the transition probability matrix for the vector-valued variable Φ_k; (2) the computational complexity will be increased drastically with the increased number of the modes by using the vector-valued jumping parameter. The proposed control schemes in this paper take advantage of the multi-step jump of Markov chains instead of augmenting the delay values.

In our previous work (Shi & Yu, 2009), the stabilization problem for the system in (6) (without considering the uncertainties and external disturbances) was solved. In the literature, to incorporate the control performance indices ($\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms) into the two-mode-dependent controller design for NCSs has not been investigated despite their importance in practical applications. The objective of this paper is to develop the robust $\mathcal{H}_2$ and robust mixed $\mathcal{H}_2/\mathcal{H}_\infty$ two-mode-dependent control design schemes for NCSs.

3. Two-mode-dependent robust $\mathcal{H}_2$ control and robust mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control

3.1. Robust stability analysis

First of all, let us deal with the robust stability analysis problem. The stochastic stability for the system in (6) with $\omega(k) = 0$ is defined in Shi and Yu (2009) considering the multi-jump and interdependency of these stochastic variables τ_k, d_k, and $d_{k-\tau_k-1}$. The following theorem provides the sufficient and necessary condition under which the closed-loop system in (6) with a designed controller and $\omega(k) = 0$ is robustly stable.

Theorem 1. *Let the controller parameters $F(i, r)$, $G(i, r)$, $H(i, r)$, and $T(i, r)$ in (3) be given and the norm-bounded uncertainty condition (2) hold. Then, the closed-loop system in (6) with $\omega(k) = 0$ is stochastically stable if and only if there exist symmetric $P(i, r) > 0$ such that the following matrix inequality:*

$$L(i,r) = -P(i,r) + \sum_{j=0}^{\tau}\sum_{s_2=0}^{d}\sum_{s_1=0}^{d} \Lambda_{ij}\Pi_{rs_2}^{1+i-j}\Pi_{s_2 s_1}^{j} \times \left[\bar{A} + \bar{B}K(i,r,s_1)\bar{C}(i)\right]^{\mathrm{T}} P(j, s_2) \times \left[\bar{A} + \bar{B}K(i,r,s_1)\bar{C}(i)\right] < 0 \quad (8)$$

holds for all $i \in \mathcal{M}$ and $r \in \mathcal{N}$. Here, $\Pi^{1+i-j}_{rs_2}$ stands for the probability of the jump from r to s_2 based on the transition probability matrix Π^{1+i-j} and $\Pi^j_{s_2 s_1}$ represents the probability of the jump from s_2 to s_1 based on the transition probability matrix Π^j.

Proof. The proof can be arrived at by following a similar line in Shi and Yu (2009). □

3.2. Definitions of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms

As the closed-loop system in (6) under consideration is a special discrete-time jump linear system, the definitions of the classical $\mathcal{H}_2$ norm and the MJLS $\mathcal{H}_2$ norm are not suitable for this system. Therefore, we define the generalized $\mathcal{H}_2$ norm for the system in (6), taking the special features of the system in (6) into account.

Definition 1. The $\mathcal{H}_2$ norm of the system in (6) is defined as

$$\|H_{y\omega}\|_2^2 = \sum_{s=1}^{l}\sum_{i_o=0}^{\tau}\sum_{r_o=0}^{d} \alpha_{(i_o,r_o)} \|y_{s,i_o,r_o}\|_2^2, \tag{9}$$

where y_{s,i_o,r_o} is the output sequence of the system in (6) and $\|y_{s,i_o,r_o}\|_2^2 = \sum_{k=0}^{\infty} \mathcal{E}\left(\|y_{s,i_o,r_o}(k)\|_2^2\right)$ when

(1) the input sequence is given by $\omega = (\omega(0), \omega(1), \ldots)$, $\omega(0) = e_s$, $\omega(k) = 0$, $k > 0$, $e_s \in \mathbb{R}^l$ the unitary vector formed by one at the sth position and zeros elsewhere;
(2) $\tau_0 = i_o$ and $d_{-\tau_0-1} = r_o$;
(3) The joint probability of τ_0 and $d_{-\tau_0-1}$ is denoted as $\alpha_{(i_o,r_o)}$, which satisfies $\sum_{i_o\in\mathcal{M}, r_o\in\mathcal{N}} \alpha_{(i_o,r_o)} = 1$, where $i_o \in \mathcal{M}$, $r_o \in \mathcal{N}$.

Remark 3. When $\tau = 0, d = 0$, the generalized $\mathcal{H}_2$ norm given in Definition 1 is reduced to the classical $\mathcal{H}_2$ norm. Hence, the definition can be viewed as a generalization of the $\mathcal{H}_2$ norm from the LTI system to the special jump linear system. Moreover, when $d = 0$, Definition 1 is reduced to the $\mathcal{H}_2$ norm for MJLSs (Costa & Marques, 1998).

The definition of the classical $\mathcal{H}_\infty$ norm for LTI systems can be interpreted as a measure of robust stability that represents the worst-case energy attenuation for any energy-bounded disturbance. Following the time-domain interpretation, the generalized $\mathcal{H}_\infty$ norm for the special system in (6) is defined as follows.

Definition 2. Let $X(0) = 0$ and define the $\mathcal{H}_\infty$ norm as

$$\|H_{y\omega}\|_\infty = \sup_{\tau_0\in\mathcal{M}} \sup_{d_{-\tau_0-1}\in\mathcal{N}} \sup_{\omega\in l_2(0,\infty)} \frac{\|y\|_2}{\|\omega\|_2}. \tag{10}$$

The following theorem establishes the relationship between the $\mathcal{H}_2$ norm and the state–space model of the jump linear system in (6).

Theorem 2. *The $\mathcal{H}_2$ norm of the system in (6) with the stochastically stabilizing controller in (3) can be computed as follows.*

$$\|H_{y\omega}\|_2^2 = \sum_{i_o=0}^{\tau}\sum_{r_o=0}^{d}\sum_{j_o=0}^{\tau}\sum_{s_{o2}=0}^{d} \alpha_{(i_o,r_o)} \Lambda_{i_o j_o} \Pi^{1+i_o-j_o}_{r_o s_{o2}} \times \mathrm{tr}\left\{\tilde{J}^{\mathrm{T}} S(j_o, s_{o2}) \tilde{J}\right\}, \tag{11}$$

where $S(j_o, s_{o2}) > 0$ is obtained from the following equation

$$S(i,r) = \tilde{C}^{\mathrm{T}}\tilde{C} + \sum_{j=0}^{\tau}\sum_{s_1=0}^{d}\sum_{s_2=0}^{d} \Lambda_{ij}\Pi^{1+i-j}_{rs_2}\Pi^{j}_{s_2 s_1} \times \left[\bar{A} + \bar{B}K(i,r,s_1)\bar{C}(i)\right]^{\mathrm{T}} S(j,s_2) \times \left[\bar{A} + \bar{B}K(i,r,s_1)\bar{C}(i)\right], \tag{12}$$

for $i \in \mathcal{M}$, $r \in \mathcal{N}$.

Proof. Suppose that $y(k)$ is an impulse response of the system in (6). Then, for $k \geq 1$ and considering (12)

$$\begin{aligned}
&\mathcal{E}\left[y(k)^{\mathrm{T}}y(k)\right] \qquad (13)\\
&= \mathcal{E}\left[X(k)^{\mathrm{T}}\tilde{C}^{\mathrm{T}}\tilde{C}X(k)\right]\\
&= \mathcal{E}\left\{X(k)^{\mathrm{T}}\left[S(i,r) - \sum_{j=0}^{\tau}\sum_{s_1=0}^{d}\sum_{s_2=0}^{d}\Lambda_{ij}\Pi^{1+i-j}_{rs_2}\Pi^{j}_{s_2s_1} \times \left(\bar{A} + \bar{B}K(i,r,s_1)\bar{C}(i)\right)^{\mathrm{T}} S(j,s_2) \times \left(\bar{A} + \bar{B}K(i,r,s_1)\bar{C}(i)\right)\right]X(k)\right\}\\
&= \mathcal{E}\left\{X(k)^{\mathrm{T}}S(i,r)X(k)\right\} - \mathcal{E}\left\{X(k+1)^{\mathrm{T}}S(\tau_{k+1}, d_{k-\tau_{k+1}})X(k+1)\right\}. \qquad (14)
\end{aligned}$$

Here, $i = \tau_k$, $r = d_{k-\tau_k-1}$.

Eq. (14) is obtained by considering the multi-step delay mode jump and the probability transition matrices:

$$\tau_k \to \tau_{k+1} : \Lambda, \qquad d_{k-i-1} \to d_{k-j} : \Pi^{1+i-j}, \qquad d_{k-j} \to d_k : \Pi^j.$$

The detailed derivation of (14) can be found in Shi and Yu (2009). Also notice that $S(i,r)$ in (12) satisfies inequalities (8). Hence, the system in (6) is stochastically stable. Then stochastic stability implies $\lim_{k\to\infty} \mathcal{E}(\|X(k)\|)^2 = 0$. By taking the sum of (13) from 1 to ∞, we obtain

$$\begin{aligned}
&\|(y_{s,i_o,j_o})\|_2^2\\
&= \sum_{k=1}^{\infty} \mathcal{E}\left\|\left[y_{s,i_o,j_o}(k)\right]\right\|^2 |_{\tau_0, d_{-\tau_0-1}}\\
&= \mathcal{E}\left\{X(1)^{\mathrm{T}}S(\tau_1, d_{-\tau_1})X(1)|_{\tau_0, d_{-\tau_0-1}}\right\}\\
&= \mathcal{E}\left\{e_s^{\mathrm{T}}\tilde{J}^{\mathrm{T}}S(\tau_1, d_{-\tau_1})\tilde{J}e_s\right\}\\
&= \sum_{j_o=0}^{\tau}\sum_{s_{o2}=0}^{d} \Lambda_{i_o j_o}\Pi^{1+i_o-j_o}_{r_o s_{o2}}\left\{e_s^{\mathrm{T}}\tilde{J}^{\mathrm{T}}S(j_o, s_{o2})\tilde{J}e_s\right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|H_{y\omega}\|_2^2 &= \sum_{s=1}^{l}\sum_{i_o=0}^{\tau}\sum_{r_o=0}^{d} \alpha_{(i_o,r_o)}\|(y_{s,i_o,j_o})\|_2^2\\
&= \sum_{i_o=0}^{\tau}\sum_{r_o=0}^{d}\sum_{j_o=0}^{\tau}\sum_{s_{o2}=0}^{d} \alpha_{(i_o,r_o)}\Lambda_{i_o j_o}\Pi^{1+i_o-j_o}_{r_o s_{o2}} \times \mathrm{tr}\left\{\tilde{J}^{\mathrm{T}}S(j_o, s_{o2})\tilde{J}\right\}.
\end{aligned}$$

This completes the proof. □

3.3. Robust $\mathcal{H}_2$ control

In this section, we focus on the two-mode-dependent robust $\mathcal{H}_2$ control design for the special jump system. The objective is to design a controller in (3) such that the $\mathcal{H}_2$ norm of the system in (6) is minimized. The $\mathcal{H}_2$ control for LTI systems has been studied (Oliveira, Geromel, & Bernussou, 2002; Scherer, Gahinet, & Chilali, 1997) and the $\mathcal{H}_2$ control for MJLSs has been investigated in Costa and Marques (2000). Based on the proposed definition of $\mathcal{H}_2$ norm for the special jump system, the $\mathcal{H}_2$ control design will be transformed to solving an optimization problem.

Theorem 3. *Under the proposed output feedback control law (3), the closed-loop system in (6) is stochastically stable and $\|H_{y\omega}\|_2 < \gamma$,* *if there exist matrices $F(i,r), G(i,r), H(i,r), T(i,r)$, and symmetric matrices $\bar{X}(j,s_2) > 0, P(i,r) > 0$ and a set of scalars $\varepsilon_1(i,r) > 0$, $\varepsilon_2(i,r) > 0, \ldots, \varepsilon_{(\tau+1)(d+1)(d+1)}(i,r) > 0$ satisfying the following inequalities*

$$\sum_{i_o=0}^{\tau}\sum_{r_o=0}^{d}\sum_{j_o=0}^{\tau}\sum_{s_{o2}=0}^{d}\alpha_{(i_o,r_o)}\Lambda_{i_oj_o}\Pi_{r_os_{o2}}^{1+i_o-j_o} \times \mathrm{tr}\left\{\tilde{J}^{\mathrm{T}}P(j_o,s_{o2})\tilde{J}\right\} < \gamma^2$$

$$\begin{bmatrix} -P(i,r)+\tilde{C}^{\mathrm{T}}\tilde{C} & * & * \\ Vc(i,r) & -X(i,r)+\hat{G}(i,r) & * \\ \Delta Vu(i,r) & 0 & -\hat{\varepsilon}(i,r)I \end{bmatrix} < 0;$$

$$\bar{X}(j,s_2)P(j,s_2) = I, \tag{15}$$

with

$$\hat{G}(i,r) = \mathrm{diag}\left\{\varepsilon_1(i,r)\tilde{G}_1\tilde{G}_1^{\mathrm{T}}\ \varepsilon_2(i,r)\tilde{G}_1\tilde{G}_1^{\mathrm{T}} \cdots \varepsilon_{(\tau+1)(d+1)(d+1)}(i,r)\tilde{G}_1\tilde{G}_1^{\mathrm{T}}\right\},$$

$$\hat{\varepsilon}(i,r) = \mathrm{diag}\{\varepsilon_1(i,r)\ \varepsilon_2(i,r) \cdots \varepsilon_{(\tau+1)(d+1)(d+1)}(i,r)\},$$

$$Vc(i,r) = \left[Vc_0(i,r)^{\mathrm{T}}\quad Vc_1(i,r)^{\mathrm{T}}\quad \cdots\quad Vc_\tau(i,r)^{\mathrm{T}}\right]^{\mathrm{T}},$$

$$Vc_j(i,r) = \left[Vc_{j,0}(i,r)^{\mathrm{T}}\quad Vc_{j,1}(i,r)^{\mathrm{T}}\quad \cdots\quad Vc_{j,d}(i,r)^{\mathrm{T}}\right]^{\mathrm{T}},$$

$$Vc_{j,s_2}(i,r) = \begin{bmatrix} (\Lambda_{ij}\Pi_{rs_2}^{1+i-j}\Pi_{s_20}^{j})^{\frac{1}{2}}[A_{\mathrm{Aug}}+B_{\mathrm{Aug}}K(i,r,0)\bar{C}(i)] \\ (\Lambda_{ij}\Pi_{rs_2}^{1+i-j}\Pi_{s_21}^{j})^{\frac{1}{2}}[A_{\mathrm{Aug}}+B_{\mathrm{Aug}}K(i,r,1)\bar{C}(i)] \\ \vdots \\ (\Lambda_{ij}\Pi_{rs_2}^{1+i-j}\Pi_{s_2d}^{j})^{\frac{1}{2}}[A_{\mathrm{Aug}}+B_{\mathrm{Aug}}K(i,r,d)\bar{C}(i)] \end{bmatrix},$$

$$\Delta Vu(i,r) = \left[\Delta Vu_0(i,r)^{\mathrm{T}}\quad \Delta Vu_1(i,r)^{\mathrm{T}}\quad \cdots\quad \Delta Vu_\tau(i,r)^{\mathrm{T}}\right]^{\mathrm{T}},$$

$$\Delta Vu_j(i,r) = \left[\Delta Vu_{j,0}(i,r)^{\mathrm{T}}\quad \Delta Vu_{j,1}(i,r)^{\mathrm{T}}\quad \cdots\quad \Delta Vu_{j,d}(i,r)^{\mathrm{T}}\right]^{\mathrm{T}},$$

$$\Delta Vu_{j,s_2}(i,r) = \begin{bmatrix} (\Lambda_{ij}\Pi_{rs_2}^{1+i-j}\Pi_{s_20}^{j})^{\frac{1}{2}}[U_1I_2^{\mathrm{T}}+U_2I_3^{\mathrm{T}}K(i,r,0)\bar{C}(i)] \\ (\Lambda_{ij}\Pi_{rs_2}^{1+i-j}\Pi_{s_21}^{j})^{\frac{1}{2}}[U_1I_2^{\mathrm{T}}+U_2I_3^{\mathrm{T}}K(i,r,1)\bar{C}(i)] \\ \vdots \\ (\Lambda_{ij}\Pi_{rs_2}^{1+i-j}\Pi_{s_2d}^{j})^{\frac{1}{2}}[U_1I_2^{\mathrm{T}}+U_2I_3^{\mathrm{T}}K(i,r,d)\bar{C}(i)] \end{bmatrix},$$

$$X(i,r) = \mathrm{diag}\{X_0(i,r)\ X_1(i,r) \cdots X_\tau(i,r)\},$$

$$X_j(i,r) = \mathrm{diag}\{X_{j,0}(i,r)\ X_{j,1}(i,r) \cdots X_{j,d}(i,r)\},$$

$$X_{j,s_2}(i,r) = \mathrm{diag}\underbrace{\left\{\bar{X}(j,s_2)\ \bar{X}(j,s_2) \cdots \bar{X}(j,s_2)\right\}}_{d+1} \tag{16}$$

for all $i, j \in \mathcal{M}$ and $r, s_2 \in \mathcal{N}$.

Proof. This theorem can be proved using Lemma 2.4 in Xie (1996) and the Schur complement. The detailed derivation is omitted due to the limitation of length. □

3.4. Robust mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control

In this section, the two-mode-dependent robust mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem for the system in (6) is solved in terms of LMIs with nonconvex constraints. The following theorem provides the sufficient condition for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem.

Theorem 4. *If*

$$-P(i,r)+\tilde{C}^{\mathrm{T}}\tilde{C}+\sum_{j=0}^{\tau}\sum_{s_1=0}^{d}\sum_{s_2=0}^{d}\Lambda_{ij}\Pi_{rs_2}^{1+i-j}\Pi_{s_2s_1}^{j} \times\left[\bar{A}+\bar{B}K(i,r,s_1)\bar{C}(i)\right]^{\mathrm{T}}P(j,s_2)\left[\bar{A}+\bar{B}K(i,r,s_1)\bar{C}(i)\right] +\frac{1}{\gamma^2}P(i,r)\tilde{J}\tilde{J}^{\mathrm{T}}P(i,r) < 0, \tag{17}$$

then the system in (6) is stochastically stable and

- *the $\mathcal{H}_\infty$ norm of system in (6) satisfies $\|H_{y\omega}\|_\infty < \gamma$;*
- $\|H_{y\omega}\|_2^2 < \sum_{i_o=0}^{\tau}\sum_{r_o=0}^{d}\sum_{j_o=0}^{\tau}\sum_{s_{o2}=0}^{d}\alpha_{(i_o,r_o)}\Lambda_{i_oj_o}\Pi_{r_os_{o2}}^{1+i_o-j_o}\ \mathrm{tr}\left\{\tilde{J}^{\mathrm{T}}P(j_o,s_{o2})\tilde{J}\right\}$.

Proof. It is straightforward to show that (17) implies (8). By Theorem 1, we conclude that the system in (6) is stochastically stable.

Next, for any nonzero disturbance signal $\omega(k)$, it follows from (6) that

$$\begin{aligned}
&\mathcal{E}\left\{X(k+1)^{\mathrm{T}}P(\tau_{k+1},d_{k-\tau_{k+1}})X(k+1)\right\} \\
&= \mathcal{E}\left\{X(k)^{\mathrm{T}}\left[\bar{A}+\bar{B}K(i,r,s_1)\bar{C}(i)\right]^{\mathrm{T}}P(\tau_{k+1},d_{k-\tau_{k+1}}) \times\left[\bar{A}+\bar{B}K(i,r,s_1)\bar{C}(i)\right]X(k)\right\}+\mathcal{E}\left\{\omega(k)^{\mathrm{T}}\tilde{J}^{\mathrm{T}}P(\tau_{k+1},d_{k-\tau_{k+1}}) \times\left[\bar{A}+\bar{B}K(i,r,s_1)\bar{C}(i)\right]X(k)\right\} \\
&\quad+\mathcal{E}\left\{X(k)^{\mathrm{T}}\left[\bar{A}+\bar{B}K(i,r,s_1)\bar{C}(i)\right]^{\mathrm{T}}P(\tau_{k+1},d_{k-\tau_{k+1}})\tilde{J}\omega(k)\right\} \\
&\quad+\mathcal{E}\left\{\omega(k)^{\mathrm{T}}\tilde{J}^{\mathrm{T}}P(\tau_{k+1},d_{k-\tau_{k+1}})\tilde{J}\omega(k)\right\} \qquad (18) \\
&< \mathcal{E}\left\{X(k)^{\mathrm{T}}\left[P(\tau_k,d_{k-\tau_k-1})-\tilde{C}^{\mathrm{T}}\tilde{C}-\frac{1}{\gamma^2}P(i,r)\tilde{J}\tilde{J}^{\mathrm{T}}P(i,r)\right]X(k)\right\} \\
&\quad+\mathcal{E}\left\{\omega(k)^{\mathrm{T}}\tilde{J}^{\mathrm{T}}P(\tau_{k+1},d_{k-\tau_{k+1}})\left[\bar{A}+\bar{B}K(i,r,s_1)\bar{C}(i)\right]X(k)\right\} \\
&\quad+\mathcal{E}\left\{X(k)^{\mathrm{T}}\left[\bar{A}+\bar{B}K(i,r,s_1)\bar{C}(i)\right]^{\mathrm{T}}P(\tau_{k+1},d_{k-\tau_{k+1}})\tilde{J}\omega(k)\right\} \\
&\quad+\mathcal{E}\left\{\omega(k)^{\mathrm{T}}\tilde{J}^{\mathrm{T}}P(\tau_{k+1},d_{k-\tau_{k+1}})\tilde{J}\omega(k)\right\}. \qquad (19)
\end{aligned}$$

Inequality (19) follows by applying inequalities (17) to the first term of (18). Further, considering that $X(k)^{\mathrm{T}}\tilde{C}^{\mathrm{T}}\tilde{C}X(k) = \|y(k)\|_2^2$, we have the following inequality

$$\begin{aligned}
&\|P(\tau_{k+1},d_{k-\tau_{k+1}})^{\frac{1}{2}}x(k+1)\|_2^2 - \|P(\tau_k,d_{k-\tau_k-1})^{\frac{1}{2}}x(k)\|_2^2+\|y(k)\|_2^2 \qquad (20) \\
&< -\frac{1}{\gamma^2}\|\tilde{J}^{\mathrm{T}}P(\tau_k,d_{k-\tau_k-1})x(k)\|_2^2+\mathcal{E}\left\{\omega(k)^{\mathrm{T}}\tilde{J}^{\mathrm{T}}P(\tau_{k+1},d_{k-\tau_{k+1}}) \times\left[\bar{A}+\bar{B}K(i,r,s_1)\bar{C}(i)\right]X(k)\right\} \\
&\quad+\mathcal{E}\left\{X(k)^{\mathrm{T}}\left[\bar{A}+\bar{B}K(i,r,s_1)\bar{C}(i)\right]^{\mathrm{T}}P(\tau_{k+1},d_{k-\tau_{k+1}})\tilde{J}\omega(k)\right\} \\
&\quad+\mathcal{E}\left\{\omega(k)^{\mathrm{T}}\tilde{J}^{\mathrm{T}}P(\tau_{k+1},d_{k-\tau_{k+1}})\tilde{J}\omega(k)\right\} \\
&= -\frac{1}{\gamma^2}\|\tilde{J}^{\mathrm{T}}P(\tau_k,d_{k-\tau_k-1})x(k)\|_2^2 +\frac{1}{\gamma^2}\|\tilde{J}^{\mathrm{T}}P(\tau_{k+1},d_{k-\tau_{k+1}})x(k+1)\|_2^2 -\frac{1}{\gamma^2}\|\tilde{J}^{\mathrm{T}}P(\tau_{k+1},d_{k-\tau_{k+1}})x(k+1)\|_2^2 \\
&\quad+2\mathcal{E}\left\{\omega(k)^{\mathrm{T}}\tilde{J}^{\mathrm{T}}P(\tau_{k+1},d_{k-\tau_{k+1}})x(k+1)\right\} -\mathcal{E}\left\{\omega(k)^{\mathrm{T}}\tilde{J}^{\mathrm{T}}P(\tau_{k+1},d_{k-\tau_{k+1}})\tilde{J}\omega(k)\right\}. \qquad (21)
\end{aligned}$$

Rearranging the inequality above yields the following inequality

$$\begin{aligned}
&\|y(k)\|_2^2 + \|P(\tau_{k+1}, d_{k-\tau_{k+1}})^{\frac{1}{2}}x(k+1)\|_2^2 \\
&\quad - \|P(\tau_k, d_{k-\tau_k-1})^{\frac{1}{2}}x(k)\|_2^2 \\
&\quad - \frac{1}{\gamma^2}\|\tilde{J}^{\mathrm{T}}P(\tau_{k+1}, d_{k-\tau_{k+1}})x(k+1)\|_2^2 \\
&\quad + \frac{1}{\gamma^2}\|\tilde{J}^{\mathrm{T}}P(\tau_k, d_{k-\tau_k-1})x(k)\|_2^2 \\
&< -\|\frac{1}{\gamma}\tilde{J}^{\mathrm{T}}P(\tau_{k+1}, d_{k-\tau_{k+1}})x(k+1) - \gamma\omega(k)\|_2^2 \\
&\quad + \mathcal{E}\left\{\omega(k)^{\mathrm{T}}\left[\gamma^2 I - \tilde{J}^{\mathrm{T}}P(\tau_{k+1}, d_{k-\tau_{k+1}})\tilde{J}\right]\omega(k)\right\} \\
&\le \mathcal{E}\left\{\omega(k)^{\mathrm{T}}\left[\gamma^2 I - \tilde{J}^{\mathrm{T}}P(\tau_{k+1}, d_{k-\tau_{k+1}})\tilde{J}\right]\omega(k)\right\}.
\end{aligned}$$

Taking the sum from $k = 0$ to ∞, and recalling that $X(0) = 0$, $\|X(k)\|_2 \to 0$ as $k \to \infty$, we obtain

$$\begin{aligned}
\|y\|_2^2 &\le \sum_{k=0}^{\infty} \mathcal{E}\left\{\omega(k)^{\mathrm{T}}\left[\gamma^2 I - \tilde{J}^{\mathrm{T}}P(\tau_{k+1}, d_{k-\tau_{k+1}})\tilde{J}\right]\omega(k)\right\} \\
&= \gamma^2(1-\upsilon)\|\omega\|_2^2 \le \gamma^2\|\omega\|_2^2,
\end{aligned}$$

where $\upsilon \in \left(0, \frac{1}{\gamma^2}\sum_{i=0}^{\tau}\sum_{j=0}^{d}\mathrm{tr}(\tilde{J}^{\mathrm{T}}P(i,r)\tilde{J})\right)$. Thus, $\frac{\|y\|_2}{\|\omega\|_2} < \gamma$.

Further, by comparing (12) and (17), we have $P(i,r) > S(i,r)$. Hence, the inequality about $\|H_{y\omega}\|_2^2$ is verified. This completes the proof. □

Condition (17) is difficult to check because it is nonlinear. We then transform it to an equivalent condition in the form of a set of LMIs with nonconvex constraints.

Theorem 5. *Under the proposed output feedback control law (3), the closed-loop system in (6) is stochastically stable and $\|H_{y\omega}\|_2 < \beta$, $\|H_{y\omega}\|_\infty < \gamma$, if there exist matrices $F(i,r), G(i,r), H(i,r)$ and $T(i,r)$ and symmetric matrices $\bar{X}(j,s_2) > 0, P(i,r) > 0$ and a set of scalars $\varepsilon_1(i,r) > 0, \varepsilon_2(i,r) > 0, \ldots, \varepsilon_{(\tau+1)(d+1)(d+1)}(i,r) > 0$ satisfying:*

$$\sum_{i_o=0}^{\tau}\sum_{r_o=0}^{d}\sum_{j_o=0}^{\tau}\sum_{s_{o2}=0}^{d} \alpha_{(i_o,r_o)}\Lambda_{i_oj_o}\Pi_{r_os_{o2}}^{1+i_o-j_o}\mathrm{tr}\{\tilde{J}^{\mathrm{T}}P(j_o,s_{o2})\tilde{J}\} < \beta^2,$$

$$\begin{bmatrix} \tilde{C}^{\mathrm{T}}\tilde{C} - P(i,r) & * & * & * \\ Vc(i,r) & \hat{G}(i,r) - X(i,r) & * & * \\ \frac{1}{\gamma}\tilde{J}^{\mathrm{T}}P(i,r) & 0 & -I & * \\ \Delta Vu(i,r) & 0 & 0 & -\hat{\varepsilon}(i,r)I \end{bmatrix} < 0$$

$$\bar{X}(j,s_2)P(j,s_2) = I, \tag{22}$$

with the matrices defined in (16) for all $i, j \in \mathcal{M}$ and $r, s_2 \in \mathcal{N}$.

Proof. By following a similar line as in the proof of Theorem 3, this theorem can be readily proved. □

Conditions (15) and (22) contain a set of LMIs and nonconvex constraints. This can be solved by the product reduction algorithm (PRA) (Zhang, Huang, & Lam, 2003). The robust $\mathcal{H}_2$ and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problems are then transformed to the optimization problems to find the minimum values for γ and/or β using Theorems 3 and 5.

4. Numerical example

In this section, a design example on inverted pendulum systems is provided. The linearized nominal discrete-time model of the plant (sampling time $T_s = 0.05$ s) is

$$A = \begin{bmatrix} 1.01 & 0.05 \\ 0.49 & 1.01 \end{bmatrix} \qquad B = \begin{bmatrix} 0.01 \\ 0.50 \end{bmatrix}$$

$$J = \begin{bmatrix} 0.10 \\ 0.10 \end{bmatrix} \qquad C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\mathrm{T}}.$$

The eigenvalues of A are 0.7312 and 1.3676. Therefore, the nominal model is unstable. The uncertainty is unavoidable in practical systems due to inadequate knowledge of the value of model parameters and variations of the values during operation. Here, the norm-bounded uncertainties can be characterized by the following matrices

$$G_1 = \begin{bmatrix} 0.01 \\ 0.5 \end{bmatrix}, \qquad U_1 = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix}, \qquad U_2 = 0.1.$$

The random delays involved in this NCS are assumed to be $\tau_k \in \{0, 1, 2\}$ and $d_k \in \{0, 1\}$, and their transition probability matrices are given by

$$\Lambda = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.3 & 0.6 & 0.1 \\ 0.3 & 0.6 & 0.1 \end{bmatrix}, \qquad \Pi = \begin{bmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{bmatrix}.$$

The initial distribution for $(\tau_0,\ d_{-\tau_0-1})$ is equal for every $(\alpha_{(i_o,r_o)})$, where $i_o \in \mathcal{M}, r_o \in \mathcal{N}$, which means that $\alpha_{(i_o,r_o)} = \frac{1}{6}$ in the following examples.

We then consider the robust mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control for the inverted pendulum system in a simulated network environment. γ is pre-set to be 2. By searching for the optimal value for β, we obtain that the minimum value of $\mathcal{H}_2$ norm $\beta_{\min}$ is 0.385 and the corresponding system matrices of the two-mode-dependent output feedback controller including a set of matrices (omitted to save space).

To illustrate the performance of the design methods, select a set of disturbance signals as follows:

$$\omega(k) = \begin{cases} 1, & \text{for } 1 \le k \le 10, \\ -1, & \text{for } 21 \le k \le 30, \\ 0, & \text{otherwise.} \end{cases}$$

In the simulation, we assume that $\Delta_u = \sin(t)$, and it can be seen that $\Delta_u^{\mathrm{T}}\Delta_u \le 1$. It is observed that the system is stabilized through the simulation. By calculation, we have $\|\omega\|_2 = 4.4721$, $\|y\|_2 = 1.7670$ for this example, which yields

$$\frac{\|y\|_2}{\|\omega\|_2} = 0.3951 < \gamma = 2.$$

The l_2 norm of the impulse response according to Definition 1 is evaluated as

$$\sqrt{\sum_{s=1}^{l}\sum_{i_o=0}^{\tau}\sum_{r_o=0}^{d}\alpha_{(i_o,r_o)}\|y_{s,i_o,r_o}\|_2^2} = 0.2877 < 0.385.$$

These results show that the closed-loop system is stochastically stable and the robust mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control performance is satisfied.

5. Conclusion

In this paper, we study the robust control synthesis problem in NCSs with norm-bounded uncertainties. The time delays from S–C and C–A links are modeled as two Markov chains. Then, a two-mode-dependent output feedback controller design method is proposed. The controller is dependent on both S–C and C–A delays. Further, the closed-loop system is formulated as a special jump linear system. The stochastic stability analysis is addressed and the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms for this special system are defined. The $\mathcal{H}_2$ and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problems are solved in the form of a set of LMIs with nonconvex constraints, which can be efficiently solved by PRA.

The results in this paper add to the growing literature on the controller design for NCSs. The two-mode-dependent framework provided in this paper motivates analysis and synthesis of a number of interesting questions that are under investigation.

- Our model assumes that the NCS is in discrete time. Practically, an NCS is a sampled-data system where the controller is in discrete time and the plant is in continuous time. One interesting thrust of research is to develop the dynamic two-mode-dependent discrete-time controller for the continuous-time plant under the network environment. Such a problem will necessitate a different line of analysis. It would be also interesting to study the data disorder and quantization issues under the two-mode-dependent framework.
- We assume that the probability transition matrix of the Markov chain is known. Studying the effects of uncertainties and unknown parameters on the probability transition matrix is also an interesting research area.

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