International Journal of Non-Linear Mechanics 46 (2011) 795–799

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

A practical method for the evaluation of eigenfunctions from compound matrix variables in finite elastic bifurcation problems

D.M. Haughton

Department of Mathematics, University of Glasgow, Glasgow G12 8QW, UK

ARTICLE INFO

Article history:
Received 18 November 2010
Accepted 10 March 2011
Available online 21 March 2011

Keywords:
Compound matrices
Eigenfunctions
Stability
Bifurcation
Non-linear elasticity

ABSTRACT

We show how the compound matrix method can be extended to give eigenfunctions as well as generalised eigenvalues to bifurcation problems in non-linear elasticity. When the incremental problem is formulated in terms of displacements only there are significant difficulties that arise from the non-trivial boundary conditions. In order to avoid these problems we adopt a Stroh formulation of the incremental problem. This then produces trivial boundary conditions for the compound matrix eigenvalue problem and more importantly known initial conditions for the compound matrix eigenfunction problem. This results in a straightforward and robust calculation for the eigenfunctions.

1. Introduction

In an earlier attempt at this problem [1,2] it was shown how the compound matrix method could be extended to give eigenfunctions as well as eigenvalues for fourth and sixth order problems in solid mechanics and in particular bifurcation problems in non-linear elasticity. The method for fourth order problems involved solving a second order system with one known initial condition (found simply by a normalisation condition) and one unknown condition. A simple shooting method to satisfy a target condition worked very well. For the sixth order problem the derivation of the eigenfunctions required the solution of a third order system with one normalised initial condition and two unknown initial conditions. We again required a shooting method to achieve a given target condition at the other end of the range. While this method could be made to work and give reasonable solutions there was a considerable effort required in finding initial approximations to the unknown initial conditions, so much so that the method could not really be recommended as a practical proposition.

It has been found that the trivial boundary conditions naturally associated with similar problems in fluid mechanics lead to a much simpler calculation for the eigenfunctions, see [3–5]. With this in mind, we re-examine the solid mechanics bifurcation problem but we now focus attention on the boundary conditions for the original eigenvalue problem. We show that adopting a Stroh formulation of the bifurcation problem, see [6], and references therein, for example, leads to trivial initial conditions and a trivial target condition for the standard compound matrix eigenvalue problem. This is of no real consequence for the eigenvalue problem, however, it does lead to a different approach to the eigenfunction problem. To determine the three components of the incremental displacements we now have to solve a sixth order initial value problem with known initial conditions rather than the third order system with shooting for two unknowns that we had in [2]. In this case we also solve for the (possibly unwanted) incremental stresses. One novel feature of the present method is that we incorporate the coefficients from the original equations whereas other approaches to the eigenfunction problem exclusively use the compound matrix variables. For some problems of interest a single incremental displacement is all that is required. In this case we show how we may isolate a single displacement and solve a third order system for the normalised displacement (and two stresses). For comparison the determinantal method (see [2]) solves a third order system three times and then solves a standard eigenvalue and eigenvector problem for a three by three matrix to obtain all three incremental displacements.

In Section 2 we demonstrate the Stroh formulation of a typical bifurcation problem from non-linear elasticity and how the basic compound matrix problem is then formulated. Following this we describe the compound matrix eigenvalue method and apply it to a particular problem. Throughout we make comparisons with the previous attempt at this problem [2]. We employ cylindrical coordinates in anticipation of the specific example to be used but the underlying method is clear.

2. Incremental problem and the compound matrix method

In the absence of body forces the incremental equilibrium equations can be written

$$\operatorname{div} \dot{\mathbf{s}}_0 = \mathbf{0}, \tag{1}$$

E-mail address: dmh@maths.gla.ac.uk

0020-7462/$ - see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijnonlinmec.2011.03.002

where div is the divergence operator in the current configuration and $\dot{\mathbf{s}}_0$ is the increment in the nominal stress referred to the current configuration. Henceforth $\dot{\mathbf{x}}$ will denote an increment in the quantity $\mathbf{x}$ and the subscript zero denotes evaluation in the current configuration. Since no extra loading is imposed on the surface of the body the incremental boundary conditions are given by

$$\dot{\mathbf{s}}_0^T\mathbf{n} = \mathbf{0}, \tag{2}$$

where $\mathbf{n}$ is a unit outward normal in the current configuration and a superscript T indicated the transpose. The incremental constitutive law can be written

$$\dot{\mathbf{s}}_0 = \mathbf{B}\dot{\mathbf{F}}_0^T, \tag{3}$$

where $\mathbf{B}$ is the fourth order tensor of instantaneous moduli in the current configuration and the increment of the deformation tensor $\dot{\mathbf{F}}_0$ is given by

$$\dot{\mathbf{F}}_0 = \frac{\partial \dot{\mathbf{x}}}{\partial \mathbf{x}}. \tag{4}$$

The non-zero components of $\mathbf{B}$, written in terms of the principal Cauchy stresses and the strain-energy function W, are

$$\left.\begin{aligned} &B_{iijj} = B_{jjii} = \frac{\lambda_i\lambda_j}{J}\frac{\partial^2 W}{\partial\lambda_i\partial\lambda_j}, \\ &B_{ijij} = \lambda_i^2\frac{\sigma_i-\sigma_j}{\lambda_i^2-\lambda_j^2}, \quad i\neq j,\ \lambda_i\neq\lambda_j, \\ &B_{ijij} = (B_{iiii}-B_{iijj}+\sigma_i)/2, \quad i\neq j,\ \lambda_i=\lambda_j, \\ &B_{ijij}-B_{ijji} = B_{ijij}-B_{jiij} = \sigma_i, \quad i\neq j, \end{aligned}\right\} \tag{5}$$

where σ_i are the principal values of the Cauchy stress tensor given by

$$\sigma_i = J^{-1}\lambda_i W_i, \tag{6}$$

λ_i are the principal stretches, the positive eigenvalues of $\mathbf{U}$ where $\mathbf{U}^2 = \mathbf{F}^T\mathbf{F}$ and $J = \det(\mathbf{F}) > 0$ is the dilatation. See Ogden [7], for example, for further details.

Using cylindrical base vectors the displacement increment is given by the vector

$$\dot{\mathbf{x}} = u(r,\theta,z)\mathbf{e}_r + v(r,\theta,z)\mathbf{e}_\theta + w(r,\theta,z)\mathbf{e}_z, \tag{7}$$

and the corresponding matrix of the increment of the deformation tensor has components

$$\dot{\mathbf{F}}_0 = \begin{bmatrix} u_r & (u_\theta - v)/r & u_z \\ v_r & (u+v_\theta)/r & v_z \\ w_r & w_\theta/r & w_z \end{bmatrix}, \tag{8}$$

with respect to cylindrical coordinates, where subscripts denote partial derivatives. In order to obtain a purely real formulation of the problem we assume, without loss of generality, that the incremental displacement functions have the form

$$u = if(r)e^{i(mt+\alpha z)},$$

$$v = g(r)e^{i(mt+\alpha z)},$$

$$w = h(r)e^{i(mt+\alpha z)}. \tag{9}$$

Instead of simply substituting (8) with (5), (9) and (3) into (1) to obtain three second order equations for f, g and h, we keep the three incremental stresses s_{rr}, $s_{r\theta}$ and s_{rz} (having dropped the superposed dot and subscript zero from the components of $\dot{\mathbf{s}}_0$) as dependent variables. The constitutive equations

$$s_{rr} = i\{B_{1111}f' + B_{1122}(f+gm)/r + B_{1133}h\alpha\}e^{i(mt+\alpha z)},$$

$$s_{r\theta} = \{B_{1212}g' - B_{1221}(mf+g)/r\}e^{i(mt+\alpha z)},$$

$$s_{rz} = \{B_{1313}h' - B_{1331}f\alpha\}e^{i(mt+\alpha z)}, \tag{10}$$

which are obtained from (3), (8) and (9), are then used to provide three equations for f', g', h'. The incremental equilibrium equations (1) can then be written as

$$\begin{aligned} &r^2 s_{rr}' + rs_{rr} + (2B_{1212} - B_{1111})rf' - (B_{3131}\alpha^2 r^2 + B_{1212}m^2 + B_{1111})f \\ &\quad + B_{1212}mrg' - m(B_{1111}+B_{1212})g - \alpha r(B_{1133}h - B_{1313}rh') = 0, \end{aligned}$$

$$\begin{aligned} &m(B_{1111} - 2B_{1212})rf' + m(B_{1111}+B_{1212})f - B_{1212}rg' \\ &\quad + (B_{1111}m^2 + B_{3131}\alpha^2 r^2 + B_{1212})g - r^2 s_{r\theta}' + mB_{1133}hr\alpha \\ &\quad + B_{1313}rhm\alpha - rs_{r\theta} = 0, \end{aligned}$$

$$\begin{aligned} &B_{1133}\alpha r^2 f' + B_{1133}\alpha rf + m\alpha r(B_{1313}+B_{1133})g \\ &\quad + (B_{1313}m^2 + B_{3333}\alpha^2 r^2)h - r^2 s_{rz}' - rs_{rz} = 0, \end{aligned} \tag{11}$$

which provide three equations for $s_{rr}', s_{r\theta}'$ and s_{rz}'. To obtain a compound matrix eigenfunction method we consider the six first order equations, (10) and (11), for $\mathbf{y} = (f,g,h,\sigma_{rr},\sigma_{r\theta},\sigma_{rz})$ in the form

$$f' = \alpha_1 f + \alpha_2 g + \alpha_3 h + \alpha_4 s_{rr}, \tag{12}$$

$$g' = \beta_1 f + \beta_2 g + \beta_3 h + \beta_4 s_{r\theta}, \tag{13}$$

$$h' = \gamma_1 f + \gamma_2 g + \gamma_3 h + \gamma_4 s_{rz}, \tag{14}$$

$$s_{rr}' = a_1 f + a_2 g + a_3 h + a_4 s_{rr} + a_5 s_{r\theta} + a_6 s_{rz}, \tag{15}$$

$$s_{r\theta}' = b_1 f + b_2 g + b_3 h + b_4 s_{rr} + b_5 s_{r\theta} + b_6 s_{rz}, \tag{16}$$

$$s_{rz}' = c_1 f + c_2 g + c_3 h + c_4 s_{rr} + c_5 s_{r\theta} + c_6 s_{rz}, \tag{17}$$

where the prime denotes differentiation with respect to r and the coefficients α_i, β_i and γ_i, $i=1,\ldots,6$, will depend on the parameter λ, say, that we are looking for and in general on r. We also have boundary conditions

$$s_{rr} = s_{r\theta} = s_{rz} = 0, \quad r = a,b, \tag{18}$$

and this is the significant aspect of the formulation so far as the numerical method is concerned. The boundary conditions in terms of displacements can be found in [2] and are much more complicated.

We suppose that, in principle, Eqs. (12)–(17) are solved three times with three linearly independent initial conditions (at $r=a$) which ensure that the boundary conditions (18) (at $r=a$) are satisfied. The three solutions thus obtained are labelled f^i, $i=1$, 2, 3 and similarly for the other dependent variables. The full solution can then be written

$$f = C_1 f^1 + C_2 f^2 + C_3 f^3,$$

$$g = C_1 g^1 + C_2 g^2 + C_3 g^3,$$

$$h = C_1 h^1 + C_2 h^2 + C_3 h^3,$$

$$s_{rr} = C_1 s_{rr}^1 + C_2 s_{rr}^2 + C_3 s_{rr}^3,$$

$$s_{r\theta} = C_1 s_{r\theta}^1 + C_2 s_{r\theta}^2 + C_3 s_{r\theta}^3,$$

$$s_{rz} = C_1 s_{rz}^1 + C_2 s_{rz}^2 + C_3 s_{rz}^3, \tag{19}$$

where C_1, C_2 and C_3 are arbitrary constants.

For the usual compound matrix method of solution to determine the bifurcation parameter λ we now introduce 20 new compound matrix variables $\phi_i(r)$, $i=1,\ldots,20$, defined by 3×3 determinants. If we introduce the notation

$$(u,v,w) = \begin{bmatrix} u^1 & u^2 & u^3 \\ v^1 & v^2 & v^3 \\ w^1 & w^2 & w^3 \end{bmatrix}, \tag{20}$$

then the compound matrix variables are given by a symmetric permutation of the entries in $\mathbf{y}$ to give

$$\phi_1 = (f,g,h), \quad \phi_2 = (f,g,s_{rr}),$$

$$\phi_3=(f,g,s_{r\theta}),\quad \phi_4=(f,g,s_{rz}),$$

$$\phi_5=(f,h,s_{rr}),\quad \phi_6=(f,h,s_{r\theta}),$$

$$\phi_7=(f,h,s_{rz}),\quad \phi_8=(f,s_{rr},s_{r\theta}),$$

$$\phi_9=(f,s_{rr},s_{rz}),\quad \phi_{10}=(f,s_{r\theta},s_{rz}),$$

$$\phi_{11}=(g,h,s_{rr}),\quad \phi_{12}=(g,h,s_{r\theta}),$$

$$\phi_{13}=(g,h,s_{rz}),\quad \phi_{14}=(g,s_{rr},s_{r\theta}),$$

$$\phi_{15}=(g,s_{rr},s_{rz}),\quad \phi_{16}=(g,s_{r\theta},s_{rz}),$$

$$\phi_{17}=(h,s_{rr},s_{r\theta}),\quad \phi_{18}=(h,s_{rr},s_{rz}),$$

$$\phi_{19}=(h,s_{r\theta},s_{rz}),\quad \phi_{20}=(s_{rr},s_{r\theta},s_{rz}). \tag{21}$$

We now differentiate (21) and use (12)–(17) with f replaced with f^1, etc. as required, to obtain the compound matrix differential equations:

$$\phi_1'=(\alpha_1+\beta_2+\gamma_3)\phi_1+\phi_{11}\alpha_4-\phi_6\beta_4+\phi_4\gamma_4,$$

$$\phi_2'=\phi_1a_3+(\alpha_1+a_4+\beta_2)\phi_2+\phi_5\beta_3-\phi_{11}\alpha_3+\phi_3a_5-\phi_8\beta_4+\phi_4a_6,$$

$$\phi_3'=\phi_1b_3+\phi_2b_4+(\alpha_1+b_5+\beta_2)\phi_3-\phi_{14}\alpha_4-\phi_{12}\alpha_3+\phi_4b_6+\phi_6\beta_3,$$

$$\begin{aligned}\phi_4'=&\phi_1c_3+\phi_2c_4+\phi_3c_5+(\alpha_1+c_6+\beta_2)\phi_4-\phi_{15}\alpha_4\\&-\phi_{13}\alpha_3+\phi_{10}\beta_4+\phi_7\beta_3,\end{aligned}$$

$$\phi_5'=-\phi_1a_2+\phi_2\gamma_2+(\alpha_1+\gamma_3+a_4)\phi_5+\phi_{11}\alpha_2+\phi_6a_5-\phi_9\gamma_4+\phi_7a_6,$$

$$\begin{aligned}\phi_6'=&-\phi_1b_2+\phi_3\gamma_2+\phi_5b_4+(\alpha_1+\gamma_3+b_5)\phi_6\\&+\phi_{12}\alpha_2-\phi_{10}\gamma_4-\phi_{17}\alpha_4+\phi_7b_6,\end{aligned}$$

$$\phi_7'=-\phi_1c_2+\phi_4\gamma_2+\phi_5c_4+\phi_6c_5+(\alpha_1+\gamma_3+c_6)\phi_7+\phi_{13}\alpha_2-\phi_{18}\alpha_4,$$

$$\begin{aligned}\phi_8'=&-\phi_2b_2+\phi_3a_2-\phi_5b_3+\phi_6a_3+(\alpha_1+b_5+a_4)\phi_8\\&+\phi_{14}\alpha_2+\phi_{17}\alpha_3+\phi_9b_6-\phi_{10}a_6,\end{aligned}$$

$$\begin{aligned}\phi_9'=&-\phi_2c_2+\phi_4a_2-\phi_5c_3+\phi_7a_3+\phi_8c_5\\&+(\alpha_1+a_4+c_6)\phi_9+\phi_{15}\alpha_2+\phi_{18}\alpha_3+\phi_{10}a_5,\end{aligned}$$

$$\begin{aligned}\phi_{10}'=&-\phi_3c_2+\phi_4b_2-\phi_6c_3+\phi_7b_3-\phi_8c_4+\phi_9b_4+(\alpha_1+b_5+c_6)\phi_{10}\\&+\phi_{16}\alpha_2+\phi_{19}\alpha_3+\phi_{20}\alpha_4,\end{aligned}$$

$$\begin{aligned}\phi_{11}'=&\phi_1a_1-\phi_2\gamma_1+\phi_5\beta_1+(\gamma_3+a_4+\beta_2)\phi_{11}+\phi_{17}\beta_4\\&+\phi_{12}a_5-\phi_{15}\gamma_4+\phi_{13}a_6,\end{aligned}$$

$$\phi_{12}'=\phi_1b_1-\phi_3\gamma_1+\phi_6\beta_1+\phi_{11}b_4+(\beta_2+b_5+\gamma_3)\phi_{12}+\phi_{13}b_6-\phi_{16}\gamma_4,$$

$$\begin{aligned}\phi_{13}'=&\phi_1c_1-\phi_4\gamma_1+\phi_7\beta_1+\phi_{11}c_4+\phi_{12}c_5\\&+(\beta_2+\gamma_3+c_6)\phi_{13}-\phi_{19}\beta_4,\end{aligned}$$

$$\begin{aligned}\phi_{14}'=&\phi_2b_1-\phi_3a_1+\phi_8\beta_1-\phi_{11}b_3+\phi_{12}a_3\\&+(b_5+\beta_2+a_4)\phi_{14}+\phi_{17}\beta_3-\phi_{16}a_6+\phi_{15}b_6,\end{aligned}$$

$$\begin{aligned}\phi_{15}'=&\phi_2c_1-\phi_4a_1+\phi_9\beta_1-\phi_{11}c_3+\phi_{13}a_3+\phi_{14}c_5\\&+(\beta_2+c_6+a_4)\phi_{15}+\phi_{18}\beta_3+\phi_{16}a_5-\phi_{20}\beta_4,\end{aligned}$$

$$\begin{aligned}\phi_{16}'=&\phi_3c_1-\phi_4b_1+\phi_{10}\beta_1-\phi_{12}c_3+\phi_{13}b_3\\&-\phi_{14}c_4+\phi_{15}b_4+(c_6+\beta_2+b_5)\phi_{16}+\phi_{19}\beta_3,\end{aligned}$$

$$\begin{aligned}\phi_{17}'=&\phi_5b_1-\phi_6a_1+\phi_8\gamma_1+\phi_{11}b_2-\phi_{12}a_2+\phi_{14}\gamma_2\\&+(\gamma_3+b_5+a_4)\phi_{17}+\phi_{18}b_6-\phi_{19}a_6+\phi_{20}\gamma_4,\end{aligned}$$

$$\begin{aligned}\phi_{18}'=&\phi_5c_1-\phi_7a_1+\phi_9\gamma_1+\phi_{11}c_2-\phi_{13}a_2+\phi_{15}\gamma_2+\phi_{17}c_5\\&+(\gamma_3+a_4+c_6)\phi_{18}+\phi_{19}a_5,\end{aligned}$$

$$\begin{aligned}\phi_{19}'=&\phi_6c_1-\phi_7b_1+\phi_{10}\gamma_1+\phi_{12}c_2-\phi_{13}b_2+\phi_{16}\gamma_2-\phi_{17}c_4\\&+\phi_{18}b_4+(\gamma_3+b_5+c_6)\phi_{19},\end{aligned}$$

$$\begin{aligned}\phi_{20}'=&\phi_8c_1-\phi_9b_1+\phi_{10}a_1+\phi_{14}c_2-\phi_{15}b_2+\phi_{16}a_2\\&+\phi_{17}c_3-\phi_{18}b_3+\phi_{19}a_3+(b_5+c_6+a_4)\phi_{20}.\end{aligned}\tag{22}$$

Now using the boundary conditions (18) at $r=a$, we see that initial conditions for the ϕ_i's are the trivial conditions $\phi_i(a)=0$, $i=2,\ldots,20$. We arbitrarily normalise the solution by setting $\phi_1(a)=1$.

It remains to ensure that the boundary conditions at $r=b$ are satisfied. We take the solutions (19) and substitute them into the boundary conditions (18) at $r=b$. We then require the coefficient matrix for the constants C_1, C_2, C_3 to be singular for the existence of non-trivial solutions. This then leads to the requirement that a 3×3 determinant is zero. This 3×3 determinant can be written as $\phi_{20}(b)$ and setting this to be zero gives our target condition. The compound matrix method to determine the critical value of the parameter λ, contained in the coefficients of (22), is to choose λ so that the target condition is satisfied when (22) is integrated from $r=a$ with the given initial conditions.

We note that the initial conditions and target condition are now very similar to those encountered in typical fluids problems (where the fluid velocities are usually taken to be zero on the boundaries rather than the incremental stresses).

3. Compound matrix eigenfunction

Now suppose that we have found a critical value of our parameter λ. We can then arrange to obtain values of $\phi_i(x)$ for any $x\in(a,b)$. If we differentiate the formal solution (19) we have

$$f'=C_1f^{1\prime}+C_2f^{2\prime}+C_3f^{3\prime},$$

$$g'=C_1g^{1\prime}+C_2g^{2\prime}+C_3g^{3\prime},$$

$$h'=C_1h^{1\prime}+C_2h^{2\prime}+C_3h^{3\prime},$$

$$s_{rr}'=C_1s_{rr}^{1\prime}+C_2s_{rr}^{2\prime}+C_3s_{rr}^{3\prime},$$

$$s_{r\theta}'=C_1s_{r\theta}^{1\prime}+C_2s_{r\theta}^{2\prime}+C_3s_{r\theta}^{3\prime},$$

$$s_{rz}'=C_1s_{rz}^{1\prime}+C_2s_{rz}^{2\prime}+C_3s_{rz}^{3\prime}. \tag{23}$$

Our proposed method now takes three of the equations (23) to formally solve for the constants C_1, C_2 and C_3. In the problem that we are taking as our example the incremental displacement f is of most interest and so we take equations $(23)_{1,5,6}$. If we then substitute the formal solutions for C_1, C_2 and C_3 back into $(23)_{1,5,6}$ we have

$$\begin{aligned}&s_{r\theta}'\phi_{10}-(\phi_4b_2+\phi_7b_3+\phi_9b_4+\phi_{10}b_5)s_{r\theta}\\&\quad+(\phi_3b_2+\phi_6b_3+\phi_8b_4-\phi_{10}b_6)s_{rz}\\&\quad-(\phi_{10}b_1+\phi_{16}b_2+\phi_{19}b_3+\phi_{20}b_4)f=0,\end{aligned}$$

$$\begin{aligned}&s_{rz}'\phi_{10}-(\phi_4c_2+\phi_7c_3+\phi_9c_4+\phi_{10}c_5)s_{r\theta}\\&\quad+(\phi_3c_2+\phi_6c_3+\phi_8c_4-\phi_{10}c_6)s_{rz}\\&\quad-(\phi_{10}c_1+\phi_{16}c_2+\phi_{19}c_3+\phi_{20}c_4)f=0,\end{aligned}$$

$$\begin{aligned}&f'\phi_{10}-(\phi_4\alpha_2+\phi_7\alpha_3+\phi_9\alpha_4)s_{r\theta}+(\phi_3\alpha_2+\phi_6\alpha_3+\phi_8\alpha_4)s_{rz}\\&\quad-(\phi_{10}\alpha_1+\phi_{16}\alpha_2+\phi_{19}\alpha_3+\phi_{20}\alpha_4)f=0,\end{aligned}\tag{24}$$

having multiplied by ϕ_{10} which we assume is non-zero throughout the range $r\in(a,b)$, we have substituted for the derivatives $f^{1\prime}$, etc. from (12) with f replaced by f^1, etc. and then made use of the definitions of the ϕ_i. We note that ϕ_{10} is zero at $r=a$ and so we have to integrate (24) inwards from $r=b$. This inevitably involves some potential loss of accuracy but, as we see below, this does not appear to be a serious limitation. Equations (24) are then the equations that we use to determine f, $s_{r\theta}$ and s_{rz} along with a normalisation initial condition for $f(b)=1$ and the boundary conditions (2) at $r=b$. The important feature of (24) is that there are no unknown initial conditions, the calculation is perfectly straightforward. In previous versions of this method the original equations (12) are not used to calculate the eigenfunctions so the coefficients depend only on the compound matrix variables ϕ_i. Since we have introduced the original equations (at no extra computational cost) it is clear (by reversing the derivation of (24)) that a solution to (24) will also be a solution to the original problem. If it is only the displacement f that is required we do not need anything further and we simply have to solve the third order system. If, however, other displacements are required then, in a similar way, by choosing $(23)_{2,4,6}$ to solve for C_1, C_2 and C_3 we may obtain

$$
\begin{aligned}
&s_{rr}'\phi_{15}+(\phi_4a_1-\phi_{13}a_3-\phi_{15}a_4-\phi_{16}a_5)s_{rr}\\
&\quad-(\phi_2a_1-\phi_{11}a_3+\phi_{14}a_5+\phi_{15}a_6)s_{rz}\\
&\quad-(\phi_9a_1+\phi_{15}a_2+\phi_{18}a_3-\phi_{20}a_5)g=0,\\[1ex]
&s_{rz}'\phi_{15}+(\phi_4c_1-\phi_{13}c_3-\phi_{15}c_4-\phi_{16}c_5)s_{rr}\\
&\quad-(\phi_2c_1-\phi_{11}c_3+\phi_{14}c_5+\phi_{15}c_6)s_{rz}\\
&\quad-(\phi_9c_1+\phi_{15}c_2+\phi_{18}c_3-\phi_{20}c_5)g=0,\\[1ex]
&\phi_{15}g'+(\phi_4\beta_1-\phi_{13}\beta_3-\phi_{16}\beta_4)s_{rr}-(\phi_2\beta_1-\phi_{11}\beta_3+\phi_{14}\beta_4)s_{rz}\\
&\quad-(\phi_9\beta_1+\phi_{15}\beta_2+\phi_{18}\beta_3-\phi_{20}\beta_4)g=0.
\end{aligned}
\tag{25}
$$

In this case we already know $s_{rz}(r)$ from the solution to (24) and so $(25)_2$ is not required, we may solve $(25)_{1,3}$ for g and s_{rr}, we have the boundary condition (2) for $s_{rr}(b)$ but we do not know $g(b)$. This is essentially the cause of the difficulties encountered in previous attempts at this problem. However, We may use $(25)_2$ evaluated at $r=b$ along with the boundary conditions (2) to write

$$
g(b)=\frac{s_{rz}'\phi_{15}}{(\phi_9c_1+\phi_{15}c_2+\phi_{18}c_3-\phi_{20}c_5)}.
\tag{26}
$$

Now using $(24)_3$ and $f(b)=1$

$$
g(b)=\frac{(\phi_{10}c_1+\phi_{16}c_2+\phi_{19}c_3+\phi_{20}c_4)\phi_{15}}{(-\phi_9c_1-\phi_{15}c_2-\phi_{18}c_3+\phi_{20}c_5)\phi_{10}},
\tag{27}
$$

which is a known quantity. Using this as our initial condition we may simultaneously solve the fifth order system $(24)_{1,2,3}$ and $(25)_{1,3}$ to obtain f, g and all three stresses. Finally, if we also require h we choose $(23)_{2,4,6}$ to solve for C_1, C_2 and C_3 we obtain

$$
\begin{aligned}
&s_{rr}'\phi_{17}+(\phi_6a_1+\phi_{12}a_2-\phi_{17}a_4+\phi_{19}a_6)s_{rr}\\
&\quad-(\phi_5a_1+\phi_{11}a_2+\phi_{17}a_5+\phi_{18}a_6)s_{r\theta}\\
&\quad-(\phi_8a_1+\phi_{14}a_2+\phi_{17}a_3+\phi_{20}a_6)h=0,\\[1ex]
&s_{r\theta}'\phi_{17}+(\phi_6b_1+\phi_{12}b_2-\phi_{17}b_4+\phi_{19}b_6)s_{rr}\\
&\quad-(\phi_5b_1+\phi_{11}b_2+\phi_{17}b_5+\phi_{18}b_6)s_{r\theta}\\
&\quad-(\phi_{14}b_2+\phi_{17}b_3+\phi_8b_1+\phi_{20}b_6)h=0,\\[1ex]
&h'\phi_{17}+(\phi_6\gamma_1+\phi_{12}\gamma_2+\phi_{19}\gamma_4)s_{rr}-(\phi_5\gamma_1+\phi_{11}\gamma_2+\phi_{18}\gamma_4)s_{r\theta}\\
&\quad-(\phi_8\gamma_1+\phi_{14}\gamma_2+\phi_{17}\gamma_3+\phi_{20}\gamma_4)h=0,
\end{aligned}
\tag{28}
$$

but here we now need only $(28)_3$ to solve for $h(r)$ since everything else is now known. Using $(28)_1$ (or we could use $(28)_2$)

$$
h(b)=\frac{s_{rr}'\phi_{17}}{(\phi_8a_1+\phi_{14}a_2+\phi_{17}a_3+\phi_{20}a_6)}
\tag{29}
$$

and, from $(25)_1$

$$
h(b)=\frac{(\phi_9a_1+\phi_{15}a_2+\phi_{18}a_3-\phi_{20}a_5)g(b)\phi_{17}}{(\phi_8a_1+\phi_{14}a_2+\phi_{17}a_3+\phi_{20}a_6)\phi_{15}}.
\tag{30}
$$

We now have a full set of known initial conditions at $r=b$ for the sixth order system $(24)_{1,2,3}$, $(25)_{1,3}$ and $(28)_3$ to simultaneously solve for everything by integrating inwards.

The method for determining f, g and h (and also the three incremental stresses) is then to integrate equations (24), $(25)_{1,3}$ and $(28)_3$ inwards from $r=b$ with initial conditions given by (2), $f(b)=1$, (27) and (30). However, we have made the tacit assumptions that ϕ_{15} and ϕ_{17} (for Eqs. (25) and (28)) do not change sign in the interval (a,b). This may not be the case and we then have to modify our approach. For example that we consider below we find that ϕ_{15} may have a zero in the interval and so we must choose a different set of equations for g. In the above derivation of the differential equations for g we have used the equations for σ_{rr}, g and σ_{rz} to determine the constants C_1, C_2 and C_3. We can simply work our way through all of the possible combinations available to us to find alternatives. For the problem below the most favourable combination appears to be to take σ_{rr}, g and h. This then leads to the two equations

$$
\begin{aligned}
&s_{rr}'\phi_{11}-(\phi_1a_1+\phi_{11}a_4+\phi_{12}a_5+\phi_{13}a_6)s_{rr}\\
&\quad-(\phi_5a_1+\phi_{11}a_2+\phi_{17}a_5+\phi_{18}a_6)g\\
&\quad+(\phi_2a_1-\phi_{11}a_3+\phi_{14}a_5+\phi_{15}a_6)h=0,\\[1ex]
&g'\phi_{11}-(\phi_1\beta_1+\phi_{12}\beta_4)s_{rr}-(\phi_5\beta_1+\phi_{11}\beta_2+\phi_{17}\beta_4)g\\
&\quad+(\phi_2\beta_1-\phi_{11}\beta_3+\phi_{14}\beta_4)h=0.
\end{aligned}
\tag{31}
$$

This has the effect of linking the equations for g and h so that we have to calculate both. If, however, we do require all three incremental displacements there is no disadvantage.

4. Example: an elastic tube under axial compression

The purpose of this section is to show that the compound matrix analysis does lead to a reliable method for calculating the eigenfunctions that does not suffer from the problems encountered in [2]. The example that we consider is taken from [8] and has an exact solution in terms of Bessel functions (this is also the example considered in [2]). This allows us to compare results obtained in three distinct ways, the new compound matrix method outlined above, an exact solution and the determinantal method, see [1,2] for a description of this approach.

We suppose that the cylindrical tube is composed of a compressible neo-Hookean material with a strain-energy function of the specific form

$$
W=\mu(I_1-3)/2-(\kappa+\mu/3)\log(J)-(2/3\mu-\kappa)(J-1),
\tag{32}
$$

where $I_1=\lambda_1^2+\lambda_2^2+\lambda_3^2$ and $J=\lambda_1\lambda_2\lambda_3$ in terms of the principal stretches λ_i, $i=1,\dots,3$. We take the shear modulus $\mu=1$, to

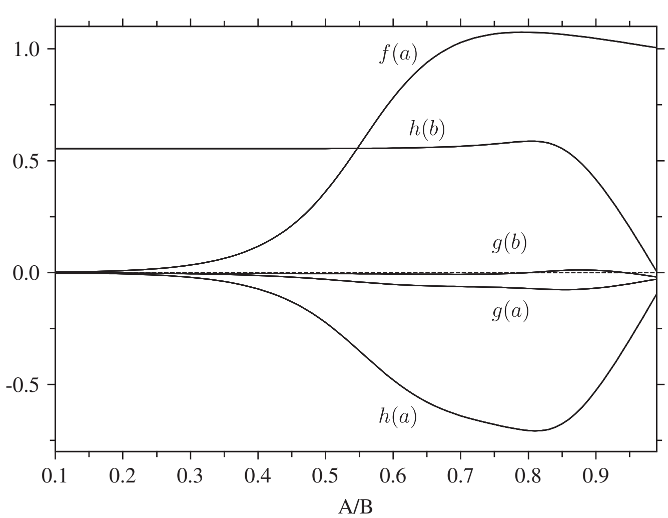

Fig. 1. A plot of $f(a)$, $g(a)$, $h(a)$, $g(b)$ and $h(b)$, which have been normalised by setting $f(b)=1$, against the undeformed shell thickness A/B. In each case three (virtually indistinguishable) curves are plotted, the exact solution and the approximate solutions found from the determinantal method and the compound matrix method.

Table 1
The absolute differences between the exact solution and the solutions obtained from the determinantal method and the compound matrix method, respectively. The mode number $m=1$ and $\kappa=5$.

$(A/B, L/B, \lambda)$	$f(a)$	$g(a)$	$h(a)$	$g(b)$	$h(b)$
(0.8, 5, 0.857156)	0.740D−09	0.104D−08	0.146D−08	0.148D−08	0.111D−08
(0.8, 5, 0.857156)	0.390D−07	0.988D−05	0.502D−04	0.627D−07	0.562D−08
(0.8, 5, 0.237732)	0.815D−08	0.113D−08	0.555D−08	0.204D−08	0.175D−07
(0.8, 5, 0.237732)	0.463D−04	0.333D−05	0.266D−07	0.598D−05	0.782D−08
(0.8, 10, 0.665951)	0.183D−08	0.586D−09	0.237D−08	0.630D−09	0.436D−08
(0.8, 10, 0.665951)	0.156D−06	0.864D−06	0.908D−05	0.431D−06	0.131D−08
(0.8, 10, 0.360244)	0.200D−07	0.504D−09	0.130D−07	0.445D−09	0.988D−08
(0.8, 10, 0.360244)	0.278D−04	0.121D−06	0.440D−06	0.394D−06	0.129D−08
(0.5, 5, 0.542021)	0.286D−07	0.170D−08	0.302D−08	0.116D−06	0.159D−06
(0.5, 5, 0.542021)	0.198D−07	0.114D−04	0.517D−04	0.399D−07	0.941D−08
(0.5, 5, 0.382195)	0.727D−07	0.140D−07	0.499D−07	0.149D−07	0.205D−07
(0.5, 5, 0.382195)	0.452D−04	0.700D−06	0.821D−05	0.102D−05	0.811D−08
(0.5, 10, 0.481038)	0.231D−06	0.187D−07	0.155D−06	0.723D−08	0.368D−07
(0.5, 10, 0.481038)	0.567D−07	0.532D−05	0.465D−04	0.624D−06	0.207D−08
(0.5, 10, 0.449951)	0.890D−04	0.724D−05	0.600D−04	0.304D−06	0.489D−06
(0.5, 10, 0.449951)	0.431D−04	0.844D−06	0.657D−05	0.451D−05	0.755D−07

normalise the equations. Values for the other parameters are what might be regarded as reasonably typical. The bulk modulus for a moderately compressible material is taken to be $\kappa=20$. The undeformed tube has a length to outer radius ratio $L/B=10$. The mode number is taken to be $m=1$.

Although this problem admits an exact solution in terms of Bessel functions we still have to evaluate the zero's of a 6×6 determinant in order to find the bifurcation parameter λ (for a given mode number m). Having found λ (and in general there will be two distinct physically reasonable values for this problem) we compute the 6×6 coefficient matrix and find the eigenvector corresponding to the smallest eigenvalue (which, ideally, will be zero) to determine the constants C_i. Since we do not have a simple direct numerical evaluation of the exact results we cannot be sure that they will be anymore accurate than either of the two other methods that we consider, see [2] for more details.

In Fig. 1 we have plotted the values determined for the three incremental displacements at the ends of the range, excluding $f(b)$ which is taken to be unity, against the increasing undeformed thickness A/B of the tube. In each case we have plotted all three solutions determined from the exact solution, the determinantal method and the present compound matrix method. Clearly the results are indistinguishable on this scale. The curves are plotted for the first critical value of λ to occur and this will also change with changing thickness A/B.

In Table 1 we give some more detailed calculations. We have taken a cylinder with a bulk modulus of $\kappa=5$ and we have chosen a number of combinations of shell thickness A/B and aspect ratio L/B. In successive rows of Table 1 we record the absolute difference between values calculated using the determinantal method and the exact solution followed by the absolute difference between values calculated using the compound matrix method and the exact solution. We note that there are two values for the bifurcation parameter λ for a given geometry.

If, for example, we look at the values in the first two rows of Table 1 it appears that the determinantal method is universally more accurate. However, if we look at the final two rows the reverse is true (apart from $g(b)$). Also, from the final two rows of Table 1 we note that the results for $f(a)$ are not as good as others for this set of parameters. The compound matrix results were calculated using the full sixth order system $(24)_{1,2,3}$, (31) and $(28)_3$. However, if we now repeat the calculation using the third order system (24) the difference between the exact and compound matrix calculations change from 0.431D−04 to 0.289D−05 while maintaining the same tolerance (5×10^{-7}) set in the differential equation solvers. (Although the smaller system also allows a smaller tolerance to be set.) The method that we have introduced here is also very flexible in the way that it can be used. Also, we could use the calculations set out above to determine the unknown initial conditions and then return to the method used in [1] to complete the calculations, although this is not an immediately attractive proposition due to the requirement of two different formulations of the basic compound matrix method.

5. Concluding remarks

We now seem to have two reliable, fast and accurate methods for determining eigenfunctions for bifurcation problems in nonlinear elasticity (the determinantal method and the compound matrix method). In practice either method is likely to give acceptable results. If there are anomalies in the results, or if an independent check is required, a comparison of the results from the two methods will be useful since the two methods are fundamentally different.

Acknowledgement

I would like to thank Michel Destrade (Paris/Dublin) for helpful conversations.

References

[1] D.M. Haughton, Evaluation of eigenfunctions from compound matrix variables in nonlinear elasticity—I. Fourth order systems, Journal of Computational Physics 227 (2008) 4478–4485.
[2] D.M. Haughton, Evaluation of eigenfunctions from compound matrix variables in nonlinear elasticity—II. Sixth order systems, Journal of Computational Physics 227 (2008) 8960–8967.
[3] B. Straughan, D.W. Walker, Two very accurate and efficient methods for computing eigenvalues and eigenfunctions in porous convection problems, Journal of Computational Physics 127 (1996) 128–141.
[4] B.S. Ng, W.H. Reid, A numerical method for linear two-point boundary-value problems using compound matrices, Journal of Computational Physics 33 (1979) 70–85.
[5] B.S. Ng, W.H. Reid, The compound matrix method for ordinary differential systems, Journal of Computational Physics 58 (1985) 209–228.
[6] R.T. Edmondson, Y.B. Fu, Stroh formulation for a generally constrained and pre-stressed elastic material, International Journal of Non-Linear Mechanics 44 (2009) 530–553.
[7] R.W. Ogden, Non-Linear Elastic Deformations, Dover Publications, New York, 1997.
[8] A. Dorfmann, D.M. Haughton, Stability and bifurcation of compressed elastic cylindrical tubes, International Journal of Engineering Science 44 (2006) 1353–1365.