On Modified Jacobi Linear Operators

J. P. Milaszewicz
Departamento de Matemática
Facultad de Ciencias Exactas y Naturales
Ciudad Universitaria
1428 Buenos Aires, Argentina

Submitted by Richard A. Brualdi

ABSTRACT

By means of successive partial substitutions, new fixed point linear equations can be obtained from old ones. The Jacobi method applied to a system in the sequence thus obtained constitutes a partial Gauss-Seidel method applied to the original one, and we analyze the behavior of the sequence of spectral radii of the successive iteration matrices (the modified Jacobi operators); we do this under the assumption that the starting operator is nonnegative with respect to a proper cone and has spectral radius less (or greater) than 1. Our main result is that, if the Jacobi operator obtained after k substitutions is irreducible, then the following one either is the same or has a strictly smaller (or greater) spectral radius. This result implies that the whole sequence of spectral radii is monotone.

1. INTRODUCTION

Set $X:=\mathbb{R}^n$, and let $B(X)$ be the space of linear mappings on X. For b in X, and L and U in $B(X)$, consider the fixed point linear equation

$$x=(L+U)(x)+b. \tag{1.1}$$

For k in $\mathbb{N}$, we define $L_k:=\sum_{j=0}^{k}L^j$ and $B_{k+1}:=LB_k+U$, with $B_0:=L+U$. Notice that if the spectral radius of L, $r(L)$, satisfies $r(L)<1$, then $\lim B_k=(I-L)^{-1}U$ (I is the identity operator), which is the Gauss-Seidel operator associated to the splitting (L,U) of B_0. It is easy to see that, if x satisfies (1.1), then we also have

$$x=B_k(x)+L_k(b). \tag{1.2}$$

The following simple lemma gives more insight into the relationship between (1.1) and (1.2) (see 2.3 in [3]).

0024-3795/83/$3.00

LEMMA 1.1. *Suppose that $I - B_0$, $I - B_k$, and L_k are invertible, and consider c in X. Then the following are equivalent*:

(i) $x = B_0(x) + b$ *and* $x = B_k(x) + c$;
(ii) x *satisfies one of the equations in* (i) *and* $L_k(b) = c$.

Consider now a proper cone K in X (see [1] for the definition); for x, y in X, we write $x \leqslant y$ if and only if $y - x \in K$; analogously, if $L, U \in B(X)$, we write $L \leqslant U$ if and only if $L(x) \leqslant U(x)$ for all x in K. Let us recall the main result of the Perron-Frobenius theory, namely, if $T \in B(X)$, $T \geqslant 0$, then there exists $x > 0$ (i.e. $x \geqslant 0$, $x \neq 0$) such that $T(x) = r(T)x$ (see [4]). Recall that a proper cone in $\mathbb{R}^n$ is always normal (see 4.1 in [2]) and that if A, B are in $B(X)$, $B \geqslant 0$, and $-B \leqslant A \leqslant B$, then $r(A) \leqslant r(B)$ (see 1.8 in [2]).

For T in $B(X)$, $T \geqslant 0$, we say that it is K-irreducible if no faces of K are invariant under T; equivalently, if $x > 0$ is such that $T(x) \leqslant ax$ for some $a \in \mathbb{R}$, then x belongs to the interior of K (denoted $x \gg 0$). If T is not K-irreducible, it is said to be K-reducible. We shall need the following extension of Theorem 9 in [4] (see also 1.3.29 in [1]).

LEMMA 1.2. *Let* $0 \leqslant A \leqslant B$, *where* B *is* K*-irreducible and* $A \neq B$. *Then* $r(A) < r(B)$.

Proof. We have $A \leqslant A + 2^{-1}(B - A) = 2^{-1}(A + B)$, which yields $r(A) \leqslant 2^{-1}r(A + B)$. Since $2^{-1}(A + B)$ is K-irreducible and $2^{-1}(A + B) \leqslant B$ with equality excluded, Theorem 9 in [4] yields $r(2^{-1}(A + B)) < r(B)$ and the conclusion follows. ∎

In the sequel L and U in $B(X)$ are such that $L \geqslant 0$, $U \geqslant 0$; for B_k as above we denote $r_k := r(B_k)$.

The following basic result will be used implicitly in this paper (See Theorem 2 in [5] and §3 in [3]): One and only one of the following holds, for all k in $\mathbb{N}$: (i) $0 = r_0 = r_k$; (ii) $0 < r_0^{k+1} \leqslant r_k \leqslant r_0 < 1$; (iii) $1 = r_0 = r_k$; (iv) $1 < r_0 \leqslant r_k \leqslant r_0^{k+1}$. Note also that, if $r(L) < 1$, then $\lim r_k = r((I - L)^{-1}U)$. F. Robert asked in [6] whether the sequence (r_k) is monotone, and the affirmative answer has been given in [3]. A further question concerns the strict monotonicity of (r_k); we analyze it in the present paper and prove in Section 3 that, if $r_0 < 1$ ($r_0 > 1$) and B_k is K-irreducible, then either $r_{k+1} < r_k$ ($r_{k+1} > r_k$) or $L^{k+1} = 0$; these results imply the monotonicity of (r_k), which is formally stated in Corollaries 3.2 and 3.4. Note that $L^{k+1} = 0$ implies that $B_{k+1} = B_k$; thus, the results already mentioned can be restated in the following way: If B_k is K-irreducible and $r_0 \neq 1$, then $r_{k+1} = r_k$ if and only if $B_{k+1} = B_k$. Some preliminary useful properties of the B_k's are proven in Section 2.

2. SOME PROPERTIES OF THE MODIFIED JACOBI OPERATORS

Recall that if B_0 is K-irreducible, then $0 < r_0$ (see Theorem 6 in [4]).

LEMMA 2.1. *Suppose B_0 is K-irreducible, $U \neq 0$, and $r_0 < 1$. Then the following hold*:

(i) $r(L^{k+1}) < r_k$.
(ii) *If $x > 0$ is such that $B_k(x) = r_k x$, then*

$$L_k^{-1}(x) = \left(I - r_k^{-1}L^{k+1}\right)^{-1} r_k^{-1}U(x) \quad \textit{and} \quad x \gg 0.$$

Proof. (i): Since $U \neq 0$, Lemma 1.2 implies that $r(L) < r(B_0)$. Thus,

$$r(L^{k+1}) = r(L)^{k+1} < r_0^{k+1} \leqslant r_k.$$

(ii): Note that $B_k = L^{k+1} + L_k U$; thus, $r_k x = L^{k+1}(x) + L_k U(x)$, which yields

$$\left(I - r_k^{-1}L^{k+1}\right)(x) = r_k^{-1}L_k U(x). \tag{2.1}$$

It follows from (i) that $I - r_k^{-1}L^{k+1}$ is invertible; this fact and the invertibility of L_k, when applied to (2.1), imply that

$$L_k^{-1}(x) = \left(I - r_k^{-1}L^{k+1}\right)^{-1} r_k^{-1}U(x).$$

As for the second part, notice that Lemma 1.1 implies

$$B_0(x) + L_k^{-1}((1 - r_k)x) = x.$$

Thus $B_0(x) \leqslant x$, which yields $x \gg 0$. ■

REMARK 2.2. It is clear from Lemma 2.1 that $r(L^{k+1}) < r_k$ is equivalent to $U \neq 0$. It is also well known (see 3.8 in [7]) that $r(L^{k+1}) < r_k$ is equivalent to $I - r_k^{-1}L^{k+1}$ being invertible and $(I - r_k^{-1}L^{k+1})^{-1} \geqslant 0$. However, one might wonder whether the hypothesis $U \neq 0$ can be dropped in the second part of 2.1(ii). The following simple example shows that it cannot: Consider $X := \mathbb{R}^2$, $K := \{(x, y) : x \geqslant 0,\ y \geqslant 0\}$, and

$$B_0 := \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix}.$$

If $U=0$, then

$$B_1=\begin{pmatrix}0.25 & 0\\ 0 & 0.25\end{pmatrix}\quad\text{and}\quad B_1\begin{pmatrix}1\\0\end{pmatrix}=0.25\begin{pmatrix}1\\0\end{pmatrix}.$$

LEMMA 2.3.

(i) *If B_k is K-irreducible, then B_0 is K-irreducible.*
(ii) *If moreover $r(L)=0$, then B_j is K-irreducible for $0\leqslant j\leqslant k$.*

Proof. (i): If we suppose that B_0 is K-reducible, there exist $t\in\mathbb{R}$, $t\geqslant 0$, and x in the boundary of K, $x\neq 0$, such that $B_0(x)=tx$. Then $t\leqslant 1$ implies that $B_j(x)\leqslant x$ for $0\leqslant j\leqslant k$, and this is a contradiction, whence $t>1$. But if $t>1$, we obtain inductively that

$$B_{j+1}(x)=LB_j(x)+U(x)\leqslant t^{j+2}x,$$

which also contradicts the K-irreducibility of B_k when $j+1=k$.

(ii): If we now suppose that for some j, $1\leqslant j\leqslant k$, B_j is K-reducible, then consider a real nonnegative t, and x in the boundary of K, $x\neq 0$, such that $B_j(x)=tx$. Suppose first that $t=0$; in this case we have $B_k(x)=0$, and this contradicts the irreducibility of B_k. Suppose now that $0<t$; recall that $L^{j+1}(x)+L_jU(x)=tx$. Thus, $t^{-1}L_jU(x)=(I-t^{-1}L^{j+1})(x)$. Since $r(L)=0$, we have that $I-t^{-1}L^{j+1}$ is invertible, and as in Lemma 2.1,

$$L_j^{-1}(x)=(I-t^{-1}L^{j+1})^{-1}t^{-1}U(x)\geqslant 0. \tag{2.2}$$

On the other hand, Lemma 1.1 implies that

$$B_0(x)+L_j^{-1}((1-t)x)=x. \tag{2.3}$$

If $t\leqslant 1$, (2.2) and (2.3) yield $B_0(x)\leqslant x$, and because of (i), it follows that $x\gg 0$. This contradiction implies that $t>1$. Note that

$$\begin{aligned}L_j^{-1}&=(I-L)(I-L^{j+1})^{-1}\\&=\left[I-L^{j+1}-L(I-L^j)\right](I-L^{j+1})^{-1}\\&=I-L(I-L^j)(I-L^{j+1})^{-1}\\&=I-LL_{j-1}L_j^{-1},\qquad\text{with}\quad L_0:=I.\end{aligned}$$

Thus, in (2.3), we get

$$B_0(x)+(t-1)LL_{j-1}L_j^{-1}(x)=tx,$$

whence $B_0(x)\leqslant tx$. This produces yet another contradiction with (i), and the proof is thus complete. ■

REMARK 2.4. The following example shows that the hypothesis $r(L)=0$ in Lemma 2.3(ii) cannot be weakened to $U\neq 0$. Consider X and K as in Remark 2.2, and

$$B_0:=\begin{bmatrix}0 & t\\ t & t\end{bmatrix},\quad L:=\begin{bmatrix}0 & t\\ t & 0\end{bmatrix}\quad \text{with}\quad 0<t.$$

Then, B_k is K-irreducible or not depending on whether k is even or odd.

3. THE STRICT MONOTONICITY QUESTION

THEOREM 3.1. *Suppose $r_0<1$. If B_k is K-irreducible, then either $r_{k+1}<r_k$ or $L^{k+1}=0$.*

Proof. Since B_k is irreducible, so is B_0, and $r_0>0$. If $U=0$, then $r_{k+1}=r_0^{k+2}<r_0^{k+1}=r_k$ for all k in $\mathbb{N}$. Suppose then that $U\neq 0$ and that $r_{k+1}=r_k$. Consider $y>0$ such that $B_{k+1}(y)=r_k y$. Equivalently

$$B_{k+1}(y)+(1-r_k)y=y.$$

By applying Lemma 1.1, we get

$$B_k(y)+L_kL_{k+1}^{-1}((1-r_k)y)=y$$

Since

$$\begin{aligned}L_kL_{k+1}^{-1}&=(I-L^{k+1})(I-L^{k+2})^{-1}\\&=\left[(I-L^{k+2})-L^{k+1}(I-L)\right](I-L^{k+2})^{-1}\\&=I-L^{k+1}L_{k+1}^{-1},\end{aligned}$$

we obtain

$$B_k(y) - L^{k+1}L_{k+1}^{-1}((1-r_k)y) = r_k y.$$

On the other hand, by applying Lemmas 2.3 and 2.1, we get

$$L_{k+1}^{-1}(y) = \left(I - r_k^{-1}L^{k+2}\right)^{-1} r_k^{-1}U(y) \geqslant 0. \tag{3.1}$$

Thus, $B_k(y) \geqslant r_k y$ and $B_k(y) \neq r_k y$ unless $L^{k+1}U(y) = 0$. But $B_k(y) \neq r_k y$ would imply $r(B_k) > r_k$ (see Theorem 10 in [3]). Hence we must have

$$L^{k+1}U(y) = 0. \tag{3.2}$$

Going back to (3.1), we get

$$L_{k+1}^{-1}(y) = r_k^{-1}U(y). \tag{3.3}$$

As $y = B_0(y) + L_{k+1}^{-1}((1-r_k)y)$, we have

$$y = B_0(y) + r_k^{-1}(1-r_k)U(y). \tag{3.4}$$

By applying L^{k+1} to both members in (3.4), and taking account of (3.2), we obtain

$$L^{k+1}(y) = L^{k+2}(y).$$

Hence, $(I-L)L^{k+1}(y) = 0$, which implies

$$L^{k+1}(y) = 0. \tag{3.5}$$

Since from Lemma 2.1 we have $y \gg 0$, (3.5) implies that $L^{k+1}(x) = 0$ for all x in K, which finally yields $L^{k+1} = 0$. ∎

COROLLARY 3.2. *Suppose $r_0 < 1$. If U is K-irreducible, then for each k, either $r_{k+1} < r_k$ or $L^{k+1} = 0$; in the latter case $r_{k+1} = r_k$. If U is K-reducible, then $r_{k+1} \leqslant r_k$ for all k (see* [3]).

Proof. The first statement follows from Theorem 3.1. As for the second, consider T in $B(X)$, $T \geqslant 0$, T K-irreducible (see 1.3 in [3]), and $t_0 \in \mathbb{R}$, $t_0 > 0$

such that $r(L+U+t_0T)<1$; for $0<t\leqslant t_0$, let us call $U(t):=U+tT$, $B_0(t):=L+U(t)$, $B_{k+1}(t):=LB_k(t)+U(t)$, and $r_k(t):=r(B_k(t))$. The first part of the present corollary implies that $r_{k+1}(t)<r_k(t)$ unless $L^{k+1}=0$; letting t tend to 0 in this inequality, we finally obtain $r_{k+1}\leqslant r_k$. ■

THEOREM 3.3. *Suppose that $r_0>1$ and B_k is K-irreducible. Then either $r_k<r_{k+1}$ or $L^{k+1}=0$.*

Proof. Evidently, we can suppose $U\neq 0$. For s and t in $\mathbb{R}$, $s>0$, $t>0$, and T as in Corollary 3.2, we define $B_0(s,t):=(L+sI)+U+tT$, $B_{m+1}(s,t):=(L+sI)B_m(s,t)+U+tT$, $L_m(s):=\sum_{j=0}^{m}(L+sI)^j$, $0\leqslant m\leqslant k$. We have thus that $r(L+sI)^{k+2}<r_{k+1}(s,t):=r(B_{k+1}(s,t))$, because $B_{k+1}(s,t)$ is K-irreducible.

Consider now sequences (s_i) and (t_i) with $s_i>0$, $t_i>0$, $\lim s_i=0=\lim t_i$, and such that $I-(L+s_iI)$, $I-(L+s_iI)^{k+2}$, $I-B_0(s_i,t_i)$, and $I-B_{k+1}(s_i,t_i)$ are invertible (see the proof of Theorem 4.1(ii) in [3] for the existence of such sequences). Consider $x_i\gg 0$ with $\|x_i\|=1$ for some fixed norm $\|\ \|$ and such that

$$B_{k+1}(s_i,t_i)(x_i)=r_{k+1}(s_i,t_i)x_i.$$

By applying Lemma 1.1 we get

$$x_i=B_0(s_i,t_i)(x_i)+[1-r_{k+1}(s_i,t_i)][L_{k+1}(s_i)]^{-1}(x_i).$$

Thus

$$x_i=B_k(s_i,t_i)(x_i)$$
$$+[1-r_{k+1}(s_i,t_i)]L_k(s_i)[L_{k+1}(s_i)]^{-1}(x_i)$$

Since from Theorem 3.1

$$L_k(s_i)[L_{k+1}(s_i)]^{-1}=I-[L(s_i)]^{k+1}[L_{k+1}(s_i)]^{-1},$$

we get

$$r_{k+1}(s_i,t_i)x_i=B_k(s_i,t_i)(x_i)$$
$$+[r_{k+1}(s_i,t_i)-1][L(s_i)]^{k+1}[L_{k+1}(s_i)]^{-1}(x_i).$$

As in the proof of Lemma 2.1, we can obtain

$$\begin{aligned}
r_{k+1}(s_i,t_i)x_i &= B_k(s_i,t_i)(x_i) \\
&\quad + [r_{k+1}(s_i,t_i)-1][L(s_i)]^{k+1}[r_{k+1}(s_i,t_i)]^{-1} \\
&\quad \times \sum_{j\geq 0} \frac{[L(s_i)]^{j(k+2)}}{[r_{k+1}(s_i,t_i)]^j} U(x_i) \\
&= B_k(s_i,t_i)(x_i) \\
&\quad + [r_{k+1}(s_i,t_i)-1][L(s_i)]^{k+1}[r_{k+1}(s_i,t_i)]^{-1} \\
&\quad \times \left(U + \sum_{j\geq 1} \frac{[L(s_i)]^{j(k+2)}}{[r_{k+1}(s_i,t_i)]^j} U \right)(x_i) \\
&\geqslant B_k(s_i,t_i)(x_i) \\
&\quad + \{[r_{k+1}(s_i,t_i)-1][L(s_i)]^{k+1}[r_{k+1}(s_i,t_i)]^{-1}U\}(x_i).
\end{aligned}$$

By considering a convergent subsequence of (x_i) we obtain $x \geqslant 0$, $\|x\|=1$, such that

$$r_{k+1}x \geqslant B_k(x) + (r_{k+1}-1)r_{k+1}^{-1}L^{k+1}U(x).$$

Thus $r_{k+1}x \geqslant B_k(x)$, with equality excluded if $L^{k+1}U(x) \neq 0$. Since B_k is K-irreducible, we must have $x \gg 0$ and $r_k < r_{k+1}$ if $L^{k+1}U(x) \neq 0$. If $L^{k+1}U(x)=0$, we get that $L^{k+1}U=0$. Since from Lemma 2.3 we have that B_0 is K-irreducible, consider $y \gg 0$ such that $B_0(y)=r_0 y$. Thus $L^{k+1}B_0(y)=L^{k+2}(y)=r_0 L^{k+1}(y)$, i.e.,

$$(r_0 I - L)L^{k+1}(y) = 0. \tag{3.6}$$

Since B_0 is K-irreducible, $r(L)<r_0$. Thus (3.6) gives $L^{k+1}(y)=0$, which implies the conclusion. ■

COROLLARY 3.4. *Suppose $r_0 > 1$. If U is K-irreducible, then either $r_k < r_{k+1}$ or $L^{k+1}=0$, for each k. If U is K-reducible, then $r_k \leqslant r_{k+1}$ for all k (see* [3]).

Proof. This follows the same lines as for Corollary 3.2 ■

Consider now $X: = \mathbb{R}^4$ and $K: = \{(x_1, x_2, x_3, x_4);\ x_i \geqslant 0,\ 1 \leqslant i \leqslant 4\}$. Let

$$L: = \begin{bmatrix} 0 & 0 & 0 & 0 \\ t & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \end{bmatrix} \quad \text{and} \quad U: = \begin{bmatrix} 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with $0 < t$. Then $B_0: = L + U$ is irreducible and

$$B_1 = \begin{bmatrix} 0 & 0 & 0 & t \\ 0 & 0 & 0 & t^2 \\ t^2 & 0 & 0 & 0 \\ 0 & t^2 & 0 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & 0 & t \\ 0 & 0 & 0 & t^2 \\ 0 & 0 & 0 & t^3 \\ t^3 & 0 & 0 & 0 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 0 & 0 & 0 & t \\ 0 & 0 & 0 & t^2 \\ 0 & 0 & 0 & t^3 \\ 0 & 0 & 0 & t^4 \end{bmatrix}.$$

Since $B_0^4 = t^4 I$, we have $r_0 = t$. By considering appropriate permutations it is easy to see that $r_1 = t^2$, $r_2 = t^2$, and $r_3 = t^4$. Thus we have

$$r_3 < r_2 = r_1 < r_0 \quad \text{if } t < 1 \quad \text{and} \quad r_0 < r_1 = r_2 < r_3 \quad \text{if } t > 1.$$

This example shows that in Theorems 3.1 and 3.3, we cannot shift the hypothesis of being K-irreducible from B_k to B_0, even when $r(L) = 0$.

REMARK 3.5. Under the assumptions that $r(L) = 0$, B_k is K-irreducible, $L^{k+1} \neq 0$, and $r_0 < 1$ ($r_0 > 1$), then Lemma 2.3(ii) and Theorem 3.1 (3.3) imply that $r_{j+1} < r_j$ ($r_{j+1} > r_j$) for all $0 \leqslant j \leqslant k$.

This work was written while the author was holding a two-month visiting position at the Scuola Normale Superiore of Pisa with the support of the Italian Consiglio Nazionale delle Ricerche.

REFERENCES

1 A. Berman and R. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic, New York, 1979.

2 J. P. Milaszewicz, On criticality and the Stein-Rosenberg theorem, *SIAM J. Numer. Anal.* 18:559–564 (1981).
3 J. P. Milaszewicz, On fixed point linear equations, *Numer. Math.* 38:53–59 (1981).
4 W. Rheinboldt and J. Vandergraft, A simple approach to the Perron-Frobenius theory for positive operators in general partially-ordered finite-dimensional linear spaces, *Math. Comp.* 27:139–145 (1973).
5 F. Robert, Algorithmes tronqués de découpe linéaire, RAIRO, revue de l'AFCET, Jul. 1972, pp. 45–64.
6 F. Robert, Autour du théoreme de Stein-Rosenberg, *Numer. Math.* 27:133–141 (1976).
7 R. S. Varga, *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, N.J., 1962.

Received 1 March 1982; revised 30 August 1982