Computers and Mathematics with Applications 56 (2008) 2048–2053

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Convergence analysis of the preconditioned Gauss–Seidel method for H-matrices[☆]

Qingbing Liu [a,b], Guoliang Chen [a,*], Jing Cai [a,c]

[a] Department of Mathematics, East China Normal University, Shanghai 200062, PR China
[b] Department of Mathematics, Zhejiang Wanli University, Ningbo 315100, PR China
[c] School of Science, Huzhou Teachers College, Huzhou 313000, PR China

ARTICLE INFO

Article history:
Received 8 August 2007
Received in revised form 25 February 2008
Accepted 5 March 2008

Keywords:
H-matrix
Preconditioner
Preconditioned iterative method
Gauss–Seidel method
H-splitting

ABSTRACT

In 1997, Kohno et al. [Toshiyuki Kohno, Hisashi Kotakemori, Hiroshi Niki, Improving the modified Gauss–Seidel method for Z-matrices, Linear Algebra Appl. 267 (1997) 113–123] proved that the convergence rate of the preconditioned Gauss–Seidel method for irreducibly diagonally dominant Z-matrices with a preconditioner $I + S_\alpha$ is superior to that of the basic iterative method. In this paper, we present a new preconditioner $I + K_\beta$ which is different from the preconditioner given by Kohno et al. [Toshiyuki Kohno, Hisashi Kotakemori, Hiroshi Niki, Improving the modified Gauss–Seidel method for Z-matrices, Linear Algebra Appl. 267 (1997) 113–123] and prove the convergence theory about two preconditioned iterative methods when the coefficient matrix is an H-matrix. Meanwhile, two novel sufficient conditions for guaranteeing the convergence of the preconditioned iterative methods are given.

1. Introduction

We consider the following linear system

$$Ax = b, \tag{1}$$

where A is a complex $n \times n$ matrix, x and b are n-dimensional vectors. For any splitting, $A = M - N$ with the nonsingular matrix M, the basic iterative method for solving the linear system (1) is as follows:

$$x^{i+1} = M^{-1}Nx^i + M^{-1}b \quad i = 0, 1, 2, \ldots.$$

Some techniques of preconditioning which improve the rate of convergence of these iterative methods have been developed.

Let us consider a preconditioned system of (1)

$$PAx = Pb, \tag{2}$$

where P is a nonsingular matrix. The corresponding basic iterative method is given in general by

$$x^{i+1} = M_P^{-1}N_Px^i + M_P^{-1}Pb \quad i = 0, 1, 2, \ldots,$$

where $PA = M_P - N_P$ is a splitting of PA.

☆ This project was granted financial support by the Shanghai Science and Technology Committee and Shanghai Priority Academic Discipline Foundation (no. 062112065).

* Corresponding author.
E-mail addresses: lqb_2008@hotmail.com (Q. Liu), glchen@math.ecnu.edu.cn (G. Chen).

0898-1221/$ – see front matter Crown Copyright © 2008 Published by Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2008.03.033

In 1997, Kohno et al. [1] proposed a general method for improving the preconditioned Gauss–Seidel method with the preconditioned matrix $P = I + S_\alpha$, if A is a nonsingular diagonally dominant Z-matrix with some conditions, where

$$S_\alpha = \begin{bmatrix} 0 & -\alpha_1 a_{1,2} & 0 & \cdots & 0 \\ 0 & 0 & -\alpha_2 a_{2,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\alpha_{n-1} a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

They showed numerically that the preconditioned Gauss–Seidel method is superior to the original iterative method if the parameters $\alpha_i \geq 0$ $(i = 1, 2, \ldots, n-1)$ are chosen appropriately.

Many other researchers have considered left preconditioners applied to linear system (1) that made the associated Jacobi and Gauss–Seidel methods converge faster than the original ones. Such modifications or improvements based on prechosen preconditioners were considered by Milaszewicz [2] who based his ideas on previous ones (see, e.g., [3]), by Gunawardena et al. [4], and very recently by Li and Sun [5] who extended the class of matrices considered in [1] and by other researchers (see, e.g., [6–12]), and many results for more general preconditioned iterative methods were obtained.

In this paper, besides the above preconditioned method, we will consider the following preconditioned linear system

$$A_\beta x = b_\beta, \tag{3}$$

where $A_\beta = (I + K_\beta)A$ and $b_\beta = (I + K_\beta)b$ with

$$K_\beta = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ -\beta_1 a_{2,1} & 0 & \cdots & 0 & 0 \\ 0 & -\beta_2 a_{3,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\beta_{n-1} a_{n,n-1} & 0 \end{bmatrix},$$

where $\beta_i \geq 0$ $(i = 1, 2, \ldots, n-1)$. Our work gives the convergence analysis of the above two preconditioned Gauss–Seidel methods for the case when a coefficient matrix A is an H-matrix and obtains two sufficient conditions for guaranteeing the convergence of two preconditioned iterative methods.

2. Preliminaries

Without loss of generality, let the matrix A of the linear system (1) be $A = I - L - U$, where I is an identity matrix, L and U are strictly lower and upper triangular matrices obtained from A, respectively.

We assume $a_{i,i+1} \neq 0$, considering the preconditioner $P = I + S_\alpha$, then we have

$$A_\alpha = (I + S_\alpha)A = I - L - S_\alpha L - (U - S_\alpha + S_\alpha U)$$
$$b_\alpha = (I + S\alpha)b,$$

whenever

$$\alpha_i a_{i,i+1} a_{i+1,i} \neq 1 \quad \text{for } i = 1, 2, \ldots, n-1,$$

then $(I - L - S_\alpha L)^{-1}$ exists. Hence it is possible to define the Gauss–Seidel iteration matrix for A_α, namely

$$T_\alpha = (I - L - S_\alpha L)^{-1}(U - S_\alpha + S_\alpha U). \tag{4}$$

Similarly, if $a_{i,i-1} \neq 0$, considering the preconditioner $P = I + K_\beta$, then we have

$$A_\beta = (I + K_\beta)A = I - L + K_\beta - K_\beta L - (U + K_\beta U)$$
$$b_\beta = (I + K_\beta)b,$$

and define the Gauss–Seidel iteration matrix for A_β, namely

$$T_\beta = (I - L + K_\beta - K_\beta L)^{-1}(U + K_\beta U). \tag{5}$$

We first recall the following: A real vector $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ is called nonnegative(positive) and denoted by $x \geq 0$ $(x > 0)$, if $x_i \geq 0$ $(x_i > 0)$ for all i. Similarly, a real matrix $A = (a_{i,j})$ is called nonnegative and denoted by $A \geq 0$ $(A > 0)$ if $a_{i,j} \geq 0$ $(a_{i,j} > 0)$ for all i, j, the absolute value of A is denoted by $|A| = (|a_{i,j}|)$.

Definition 2.1 (*[13]*). A real matrix A is called an M-matrix if $A = sI - B$, $B \geq 0$ and $s > \rho(B)$, where $\rho(B)$ denotes the spectral radius of B.

Definition 2.2 (*[13]*). A complex matrix $A = (a_{i,j})$ is an H-matrix, if its comparison matrix $\langle A\rangle = (\bar{a}_{i,j})$ is an M-matrix, where $\bar{a}_{i,j}$ is

$$\bar{a}_{i,i} = |a_{i,i}|, \qquad \bar{a}_{i,j} = -|a_{i,j}|, \quad i \neq j.$$

Definition 2.3 (*[14]*). The splitting $A = M - N$ is called an H-splitting if $\langle M\rangle - |N|$ is an M-matrix.

Lemma 2.1 (*[14]*). *Let $A = M - N$ be a splitting. If it is an H-splitting, then A and M are H-matrices and $\rho(M^{-1}N) \leq \rho(\langle M\rangle^{-1}|N|) < 1$.*

Lemma 2.2 (*[15]*). *Let A have nonpositive off-diagonal entries. Then a real matrix A is an M-matrix if and only if there exists some positive vector $u = (u_1, \ldots, u_n)^{\mathrm{T}} > 0$ such that $Au > 0$.*

3. Convergence results

Theorem 3.1. *Let A be an H-matrix with unit diagonal elements, $A_\alpha = (I + S_\alpha)A = M_\alpha - N_\alpha$, $M_\alpha = I - L - S_\alpha L$ and $N_\alpha = U - S_\alpha + S_\alpha U$. Let $u = (u_1, \ldots, u_n)^{\mathrm{T}}$ be a positive vector such that $\langle A\rangle u > 0$. Assume that $a_{i,i+1} \neq 0$ for $i = 1, 2, \ldots, n-1$, and*

$$\alpha_i' = \frac{u_i - \sum_{j=1}^{i-1} |a_{i,j}|u_j - \sum_{j=i+2}^{n} |a_{i,j}|u_j + |a_{i,i+1}|u_{i+1}}{|a_{i,i+1}| \sum_{j=1}^{n} |a_{i+1,j}|u_j},$$

then $\alpha_i' > 1$ for $i = 1, 2, \ldots, n-1$ and for $0 \leq \alpha_i < \alpha_i'$, the splitting $A_\alpha = M_\alpha - N_\alpha$ is an H-splitting and $\rho(M_\alpha^{-1}N_\alpha) < 1$ so that the iteration (2) converges to the solution of (1).

Proof. By assumption, let a positive vector $u > 0$ satisfy $\langle A\rangle u > 0$, from the definition of $\langle A\rangle$, we have

$$u_i - \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{i,j}|u_j > 0 \quad \text{for } i = 1, 2, \ldots, n-1.$$

Therefore, we have

$$\begin{aligned} &u_i - \sum_{j=1}^{i-1} |a_{i,j}|u_j - \sum_{j=i+2}^{n} |a_{i,j}|u_j + |a_{i,i+1}|u_{i+1} - |a_{i,i+1}| \sum_{j=1}^{n} |a_{i+1,j}|u_j \\ &\quad = u_i - \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{i,j}|u_j + |a_{i,i+1}| \left(u_{i+1} - \sum_{\substack{j=1\\ j\neq i+1}}^{n} |a_{i+1,j}|u_j \right) \quad \text{for } i = 1, 2, \ldots, n-1. \end{aligned}$$

Observe that $u_i - \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{i,j}|u_j > 0$ and $u_{i+1} - \sum_{\substack{j=1\\ j\neq i+1}}^{n} |a_{i+1,j}|u_j > 0$, then we have

$$u_i - \sum_{j=1}^{i-1} |a_{i,j}|u_j - \sum_{j=i+2}^{n} |a_{i,j}|u_j + |a_{i,i+1}|u_{i+1} - |a_{i,i+1}| \sum_{j=1}^{n} |a_{i+1,j}|u_j > 0,$$

and

$$u_i - \sum_{j=1}^{i-1} |a_{i,j}|u_j - \sum_{j=i+2}^{n} |a_{i,j}|u_j + |a_{i,i+1}|u_{i+1} > |a_{i,i+1}| \sum_{j=1}^{n} |a_{i+1,j}|u_j > 0 \quad \text{for } i = 1, 2, \ldots, n-1.$$

This implies

$$\alpha_i' = \frac{u_i - \sum_{j=1}^{i-1} |a_{i,j}|u_j - \sum_{j=i+2}^{n} |a_{i,j}|u_j + |a_{i,i+1}|u_{i+1}}{|a_{i,i+1}| \sum_{j=1}^{n} |a_{i+1,j}|u_j} > 1 \quad \text{for } i = 1, 2, \ldots, n-1.$$

Hence, $\alpha_i' > 1$ for $i = 1, 2, \ldots, n-1$. □

In order to prove that $\rho(M_\alpha^{-1}N_\alpha) < 1$, we first show that $\langle M_\alpha\rangle - |N_\alpha|$ is an M-matrix. Let $[(\langle M_\alpha\rangle - |N_\alpha|)u]_i$ be the ith element in the vector $(\langle M_\alpha\rangle - |N_\alpha|)u$ for $i = 1, 2, \ldots, n-1$. Then we have

$$\begin{aligned}
[(\langle M_\alpha\rangle - |N_\alpha|)u]_i &= |1-\alpha_i a_{i,i+1}a_{i+1,i}|u_i - \sum_{j=1}^{i-1}|a_{i,j}-\alpha_i a_{i,i+1}a_{i+1,j}|u_j - \sum_{j=i+1}^{n}|a_{i,j}-\alpha_i a_{i,i+1}a_{i+1,j}|u_j \\
&\geq u_i - \alpha_i|a_{i,i+1}a_{i+1,i}|u_i - \sum_{j=1}^{i-1}|a_{i,j}|u_j - \alpha_i\sum_{j=1}^{i-1}|a_{i,i+1}a_{i+1,j}|u_j \\
&\quad - \sum_{j=i+2}^{n}|a_{i,j}|u_j - \alpha_i\sum_{j=i+2}^{n}|a_{i,i+1}a_{i+1,j}|u_j - |1-\alpha_i||a_{i,i+1}|u_{i+1},
\end{aligned} \tag{6}$$

and

$$[(\langle M_\alpha\rangle - |N_\alpha|)u]_n = u_n - \sum_{\substack{j=1\\ j\neq n}}^{n}|a_{n,j}|u_j > 0. \tag{7}$$

If $0 \leq \alpha_i \leq 1$ for $i = 1, 2, \ldots, n-1$, then we have

$$\begin{aligned}
[(\langle M_\alpha\rangle - |N_\alpha|)u]_i &\geq u_i - \alpha_i|a_{i,i+1}a_{i+1,i}|u_i - \sum_{j=1}^{i-1}|a_{i,j}|u_j - \alpha_i\sum_{j=1}^{i-1}|a_{i,i+1}a_{i+1,j}|u_j \\
&\quad - \sum_{j=i+2}^{n}|a_{i,j}|u_j - \alpha_i\sum_{j=i+2}^{n}|a_{i,i+1}a_{i+1,j}|u_j - (1-\alpha_i)|a_{i,i+1}|u_{i+1} \\
&= u_i - \sum_{\substack{j=1\\ j\neq i}}^{n}|a_{i,j}|u_j + \alpha_i|a_{i,i+1}|u_{i+1} - \alpha_i|a_{i,i+1}|\sum_{\substack{j=1\\ j\neq i+1}}^{n}|a_{i+1,j}|u_j \\
&= \left(u_i - \sum_{\substack{j=1\\ j\neq i}}^{n}|a_{i,j}|u_j\right) + \alpha_i|a_{i,i+1}|\left(u_{i+1} - \sum_{\substack{j=1\\ j\neq i+1}}^{n}|a_{i+1,j}|u_j\right).
\end{aligned} \tag{8}$$

Since $u_i - \sum_{\substack{j=1\\ j\neq i}}^{n}|a_{i,j}|u_j > 0$ and $u_{i+1} - \sum_{\substack{j=1\\ j\neq i+1}}^{n}|a_{i+1,j}|u_j > 0$, we have

$$[(\langle M_\alpha\rangle - |N_\alpha|)u]_i > 0 \quad \text{for } i = 1, 2, \ldots, n-1. \tag{9}$$

If $1 < \alpha_i < \alpha_i'$ for $i = 1, 2, \ldots, n-1$, from (6) and the definition of α_i', then we have

$$\begin{aligned}
[(\langle M_\alpha\rangle - |N_\alpha|)u]_i &\geq u_i - \alpha_i|a_{i,i+1}a_{i+1,i}|u_i - \sum_{j=1}^{i-1}|a_{i,j}|u_j - \alpha_i\sum_{j=1}^{i-1}|a_{i,i+1}a_{i+1,j}|u_j \\
&\quad - \sum_{j=i+2}^{n}|a_{i,j}|u_j - \alpha_i\sum_{j=i+2}^{n}|a_{i,i+1}a_{i+1,j}|u_j - (\alpha_i-1)|a_{i,i+1}|u_{i+1} \\
&= u_i - \sum_{j=1}^{i-1}|a_{i,j}|u_j - \sum_{j=i+2}^{n}|a_{i,j}|u_j + |a_{i,i+1}|u_{i+1} - \alpha_i|a_{i,i+1}|\sum_{j=1}^{n}|a_{i+1,j}|u_j \\
&> 0.
\end{aligned} \tag{10}$$

Therefore, from (7) to (10), we have

$$(\langle M_\alpha\rangle - |N_\alpha|)u > 0 \quad \text{for } 0 \leq \alpha_i < \alpha_i'.$$

By Lemma 2.2, $\langle M_\alpha\rangle - |N_\alpha|$ is an M-matrix for $0 \leq \alpha_i < \alpha_i'$ $(i = 1, 2, \ldots, n-1)$. From Definition 2.3, $A_\alpha = M_\alpha - N_\alpha$ is an H-splitting for $0 \leq \alpha_i < \alpha_i'$ $(i = 1, 2, \ldots, n-1)$. Hence, according to Lemma 2.1, we have that A_α and M_α are H-matrices and $\rho(M_\alpha^{-1}N_\alpha) \leq \rho(\langle M_\alpha\rangle^{-1}|N_\alpha|) < 1$ for $0 \leq \alpha_i < \alpha_i'$ $(i = 1, 2, \ldots, n-1)$.

Theorem 3.2. *Let A be an H-matrix with unit diagonal elements, $A_\beta = (I + K_\beta)A = M_\beta - N_\beta$, $M_\beta = I - L + K_\beta - K_\beta L$ and $N_\beta = U + K_\beta U$. Let $v = (v_1, \ldots, v_n)^{\mathrm{T}}$ be a positive vector such that $\langle A\rangle v > 0$. Assume that $a_{i,i-1} \neq 0$ for $i = 2, \ldots, n$, and*

$$\beta_i' = \frac{v_i - \sum_{j=1}^{i-2}|a_{i,j}|v_j - \sum_{j=i+1}^{n}|a_{i,j}|v_j + |a_{i,i-1}|v_{i-1}}{|a_{i,i-1}|\sum_{j=1}^{n}|a_{i-1,j}|v_j},$$

then $\beta_i' > 1$ for $i = 2, \ldots, n$ and for $0 \le \beta_i < \beta_i'$, the splitting $A_\beta = M_\beta - N_\beta$ is an H-splitting and $\rho(M_\beta^{-1}N_\beta) < 1$ so that the iteration (2) converges to the solution of (1).

Proof. From the conditions of the theorem, we know there exists a positive vector $v > 0$ satisfying $\langle A\rangle u > 0$, then we have

$$v_i - \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{i,j}|v_j > 0 \quad \text{for } i = 2, \ldots, n.$$

Therefore, we have

$$\begin{aligned} & v_i - \sum_{j=1}^{i-2} |a_{i,j}|v_j - \sum_{j=i+1}^{n} |a_{i,j}|v_j + |a_{i,i-1}|v_{i-1} - |a_{i,i-1}| \sum_{j=1}^{n} |a_{i-1,j}|v_j \\ &\quad = v_i - \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{i,j}|v_j + |a_{i,i-1}| \left(v_{i-1} - \sum_{\substack{j=1\\ j\neq i-1}}^{n} |a_{i-1,j}|v_j \right) \quad \text{for } i = 2, \ldots, n. \end{aligned}$$

Since $v_i - \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{i,j}|v_j > 0$ and $v_{i-1} - \sum_{\substack{j=1\\ j\neq i-1}}^{n} |a_{i-1,j}|v_j > 0$, we have

$$v_i - \sum_{j=1}^{i-2} |a_{i,j}|v_j - \sum_{j=i+1}^{n} |a_{i,j}|v_j + |a_{i,i-1}|v_{i-1} - |a_{i,i-1}| \sum_{j=1}^{n} |a_{i-1,j}|v_j > 0.$$

So it is obvious to obtain

$$v_i - \sum_{j=1}^{i-2} |a_{i,j}|v_j - \sum_{j=i+1}^{n} |a_{i,j}|v_j + |a_{i,i-1}|v_{i-1} > |a_{i,i-1}| \sum_{j=1}^{n} |a_{i-1,j}|v_j > 0 \quad \text{for } i = 2, \ldots, n.$$

This implies

$$\beta_i' = \frac{v_i - \sum\limits_{j=1}^{i-2} |a_{i,j}|v_j - \sum\limits_{j=i+1}^{n} |a_{i,j}|v_j + |a_{i,i-1}|v_{i-1}}{|a_{i,i-1}| \sum\limits_{j=1}^{n} |a_{i-1,j}|v_j} > 1 \quad \text{for } i = 2, \ldots, n.$$

Namely, $\beta_i' > 1$ for $i = 2, \ldots, n$. □

In order to prove that $\rho(M_\beta^{-1}N_\beta) < 1$, we first show that $\langle M_\beta\rangle - |N_\beta|$ is an M-matrix, let $[(\langle M_\beta\rangle - |N_\beta|)v]_i$ be the ith element in the vector $(\langle M_\beta\rangle - |N_\beta|)v$ for $i = 2, \ldots, n$. Then we have

$$\begin{aligned} [(\langle M_\beta\rangle - |N_\beta|)v]_i &= v_i - \beta_{i-1}|a_{i,i-1}a_{i-1,i}|v_i - \sum_{j=1}^{i-1} |a_{i,j} - \beta_{i-1}a_{i,i-1}a_{i-1,j}|v_j - \sum_{j=i+1}^{n} |a_{i,j} - \beta_{i-1}a_{i,i-1}a_{i-1,j}|v_j \\ &\ge v_i - \beta_{i-1}|a_{i,i-1}a_{i-1,i}|v_i - \sum_{j=1}^{i-2} |a_{i,j}|v_j - \beta_{i-1} \sum_{j=1}^{i-2} |a_{i,i-1}a_{i-1,j}|v_j \\ &\quad - \sum_{j=i+1}^{n} |a_{i,j}|v_j - \beta_{i-1} \sum_{j=i+1}^{n} |a_{i,i-1}a_{i-1,j}|v_j - |1 - \beta_{i-1}||a_{i,i-1}|v_{i-1}, \end{aligned} \tag{11}$$

and

$$[(\langle M_\beta\rangle - |N_\beta|)v]_1 = v_1 - \sum_{j=2}^{n} |a_{1,j}|v_j > 0. \tag{12}$$

If $0 \le \beta_{i-1} \le 1$ for $i = 2, \ldots, n$, then we have

$$\begin{aligned} [(\langle M_\beta\rangle - |N_\beta|)v]_i &\ge v_i - \beta_{i-1}|a_{i,i-1}a_{i-1,i}|v_i - \sum_{j=1}^{i-2} |a_{i,j}|v_j - \beta_{i-1} \sum_{j=1}^{i-2} |a_{i,i-1}a_{i-1,j}|v_j \\ &\quad - \sum_{j=i+1}^{n} |a_{i,j}|v_j - \beta_{i-1} \sum_{j=i+1}^{n} |a_{i,i-1}a_{i-1,j}|v_j - (1 - \beta_{i-1})|a_{i,i-1}|v_{i-1} \end{aligned}$$

$$= v_i - \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{i,j}|v_j + \beta_{i-1}|a_{i,i-1}|v_{i-1} - \beta_{i-1}|a_{i,i-1}| \sum_{\substack{j=1\\ j\neq i-1}}^{n} |a_{i-1,j}|v_j$$

$$= \left(v_i - \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{i,j}|v_j\right) + \beta_{i-1}|a_{i,i-1}| \left(v_{i-1} - \sum_{\substack{j=1\\ j\neq i-1}}^{n} |a_{i-1,j}|v_j\right). \quad (13)$$

Observe that $v_i - \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{i,j}|v_j > 0$ and $v_{i-1} - \sum_{\substack{j=1\\ j\neq i-1}}^{n} |a_{i-1,j}|v_j > 0$, then we have

$$[(\langle M_\beta\rangle - |N_\beta|)v]_i > 0 \quad \text{for } i = 2, 3, \ldots, n. \quad (14)$$

If $1 < \beta_{i-1} < \beta'_{i-1}$ for $i = 2, \ldots, n$, from (11) and the definition of β'_{i-1}, then we have

$$\begin{aligned}[(\langle M_\beta\rangle - |N_\beta|)v]_i &\geq v_i - \beta_{i-1}|a_{i,i-1}a_{i-1,i}|v_i - \sum_{j=1}^{i-2} |a_{i,j}|v_j - \beta_{i-1} \sum_{j=1}^{i-2} |a_{i,i-1}a_{i-1,j}|v_j \\ &\quad - \sum_{j=i+1}^{n} |a_{i,j}|v_j - \beta_{i-1} \sum_{j=i+1}^{n} |a_{i,i-1}a_{i-1,j}|v_j - (\beta_{i-1} - 1)|a_{i,i-1}|v_{i-1} \\ &= v_i - \sum_{j=1}^{i-2} |a_{i,j}|v_j - \sum_{j=i+1}^{n} |a_{i,j}|v_j + |a_{i,i-1}|v_{i-1} - \beta_{i-1}|a_{i,i-1}| \sum_{j=1}^{n} |a_{i-1,j}|v_j \\ &> 0. \end{aligned} \quad (15)$$

Therefore, from (12) to (15), we have

$$(\langle M_\beta\rangle - |N_\beta|)v > 0 \quad \text{for } 0 \leq \beta_{i-1} < \beta'_{i-1}.$$

By Lemma 2.2, $\langle M_\beta\rangle - |N_\beta|$ is an M- matrix for $0 \leq \beta_{i-1} < \beta'_{i-1}$ $(i = 2, \ldots, n)$. From Definition 2.3, $A_\beta = M_\beta - N_\beta$ is an H-splitting for $0 \leq \beta_{i-1} < \beta'_{i-1}$ $(i = 2, \ldots, n)$. Hence, from Lemma 2.1, we have that A_β and M_β are H-matrices and $\rho(M_\beta^{-1}N_\beta) \leq \rho(\langle M_\beta\rangle^{-1}|N_\beta|) < 1$ for $0 \leq \beta_{i-1} < \beta'_{i-1}$ $(i = 2, \ldots, n)$.

Remark 3.1. From Theorems 3.1 and 3.2, we observe that the two preconditioned iterative methods converge to the solution of the linear system (1) if the coefficient matrix A which is an H-matrix satisfies the conditions of the corresponding theorems. Hence the two theorems provide sufficient conditions for guaranteeing the convergence of the preconditioned iterative methods.

Acknowledgements

The authors wish to express their thanks to Professor E.Y. Rodin and the two referees for helpful suggestions and comments.

References

[1] Toshiyuki Kohno, Hisashi Kotakemori, Hiroshi Niki, Improving the modified Gauss–Seidel method for Z-matrices, Linear Algebra Appl. 267 (1997) 113–123.
[2] J.P. Milaszewicz, Improving Jacobi and Gauss–Seidel iterations, Linear Algebra Appl. 93 (1987) 161–170.
[3] J.P. Milaszewicz, On modified Jacobi linear operators, Linear Algebra Appl. 51 (1983) 127–136.
[4] A. Gunawardena, S.K. Jain, L. Snyder, Modified iterative methods for consistent linear systems, Linear Algebra Appl. 149/156 (1991) 123–143.
[5] W. Li, W.W. Sun, Modified Gauss–Seidel type methods and Jacobi type methods for Z-matrix, Linear Algebra Appl. 317 (2000) 227–240.
[6] A. Hadjidimos, D. Noutsos, M. Tzoumas, More on modifications and improvements of classical iterative schemes for M-matrices, Linear Algebra Appl. 364 (2003) 253–279.
[7] D. Noutsos, M. Tzoumas, On optimal improvements of classical iterative schemes for Z-matrix, J. Comput. Appl. Math. 188 (2006) 89–106.
[8] M. Neumann, R.J. Plemmons, Convergence of parallel multisplitting iterative methods for M-matrix, Linear Algebra Appl. 88/89 (1987) 559–573.
[9] W. Li, A note on the preconditioned Gauss–Seidel(GS) method for linear systems, J. Comput. Appl. Math. 182 (2005) 81–90.
[10] X.Z. Wang, T.Z. Huang, Y.D. Fu, Comparison results on preconditioned SOR-type method for Z-matrix linear systems, J. Comput. Appl. Math. 206 (2007) 726–732.
[11] M.J. Wu, L. Wang, Y.Z. Song, Preconditioned AOR iterative method for linear systems, Appl. Numer. Math. 57 (2007) 572–685.
[12] L.Y. Sun, Some extensions of the improved modified Gauss–Seidel iterative method for H-matrix, Numer. Linear Algebr. 13 (2006) 869–876.
[13] R.S. Varga, Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, NJ, 1962, pp. 22–23.
[14] A. Frommer, D.B. Szyld, H-splitting and two-stage iterative methods, Numer. Math. 63 (1992) 345–356.
[15] K.Y. Fan, Topological proofs for certain theorems on matrices with non-negative elements, Monatsh. Math. 62 (1958) 219–237.